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Janet bases and Gröbner bases

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JANET BASES AND GRÖBNER BASES

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1. Introduction

The results of Buchberger [3] on Gröbner bases in commutative polynomial rings have been generalized by several authors (see for example [2], [5] for early treatments) to the case of some rings of linear differential operators.

Independently, the work of Riquier [27] and Janet [13, 14] on the algebraic approach to the systems of partial differential equations was discovered by Pommaret [21, 22] (see also [26]). Since then, these works and the ideas behind them have been thoroughly explored, generalized and firmly established within the framework of effective methods for the resolution of systems of partial differential equations (see for example [28], [16], [10, 11], [30, 31], [32], [19], [25]).

Despite this, as far as we know, there is still no systematic comparison between Janet bases (called by him completely integrable systems) and Gröbner bases approach. Most references in the literature accept that both of them are "essentially equivalent".

This is the task we undertake in the present work. Concretely, we show that, under certain hypothesis, when the linear differential equations have their coefficients in a field, every completely integrable system is a Gröbner basis and conversely (see 4.1.2, 4.1.3, 4.2.5, 4.2.6). This is particularly useful in the case of rings of differential operators with constant coefficients. Being this a commutative polynomial ring, we think we can regard Janet bases as a precedent of Gröbner bases (in the commutative case).

Of course, we cannot directly apply the result of Janet to the case of differential equations with coefficients in a ring. This generalization does exists for the Gröbner bases theory, in the case of "general" rings of differential operators (see for example [2], [5, 6, 7], [15], [18], [12], [29]).

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2. Monomials

Let $\mathbf{k}$ be a field. Let denote by $\mathcal{M}(X)$ the set of monic monomials of the commutative polynomial ring $\mathbf{k}[X] = \mathbf{k}[x_1, \dots, x_n]$. If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$ we will write X^α instead of $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In ([13]; (1920)) Janet gives a proof of the so-called "Dickson's lemma" and, as a consequence, he proves ([13], pp. 69-70) the following two lemmas:

Lemma 2.1.1. *Let I be an infinite subset of $\mathcal{M}(X)$. Then there exists a finite subset $F \subset I$ such that for all $X^\alpha \in I$ there exists $X^\beta \in F$ such that X^α is divisible by X^β.*

Lemma 2.1.2. *Let $S_1 \subset S_2 \subset \cdots \subset S_i \subset \cdots$ be an increasing sequence of subsets in $\mathcal{M}(X)$ such that for all i, each monomial in $S_{i+1} \setminus S_i$ is not divisible by a monomial in S_i. Then this sequence is finite.*

2.2. Janet modules.

Definition 2.2.1. *We say that a subset J of $\mathcal{M}(X)$ is a Janet module if either $J = \emptyset$ or each multiple of a monomial in J lies in J:*

$$\forall X^\alpha \in J, \forall \beta \in \mathbf{N}^n \text{ we have } X^{\alpha+\beta} \in J.$$

Remark 2.2.2. *Let $\phi : \mathcal{M}(X) \to \mathbf{N}^n$ be the canonical map $\phi(X^\alpha) = (\alpha_1, \ldots, \alpha_n)$. $J \subset \mathcal{M}(X)$ is a Janet module if and only if $\phi(J) + \mathbf{N}^n = \phi(J)$.*

Definition 2.2.3. *Let $J \neq \emptyset$ be a Janet module. A finite subset $\mathcal{B}$ of J is said to be a basis of J if each monomial in J is divisible by some monomial in $\mathcal{B}$.*

Proposition 2.2.4. *Each Janet module has a basis.*

Proof. Apply 2.1.1. □

2.3. Multiplicative variables. Classes. Here we will give the definition (Janet, [13], pp. 75-76) of multiplicative (and non-multiplicative) variables and the definition of class of a monomial.

Definition 2.3.1. *Let $\mathcal{F}$ be a finite subset of $\mathcal{M}(X)$ and $X^\alpha \in \mathcal{F}$.*
1. x_n is said to be a multiplicative variable with respect to (or simply, for) X^α in $\mathcal{F}$ if for all $X^\beta \in \mathcal{F}$ we have $\beta_n \leq \alpha_n$.
2. x_j, $1 \leq j \leq n-1$, is said to be a multiplicative variable with respect to (or simply, for) X^α in $\mathcal{F}$ if the following condition holds: for all $X^\beta \in \mathcal{F}$ with $\beta_n = \alpha_n, \cdots, \beta_{j+1} = \alpha_{j+1}$, we have $\beta_j \leq \alpha_j$. We denote by $\mathrm{mult}(X^\alpha, \mathcal{F})$ the set of multiplicative variables with respect to X^α in $\mathcal{F}$. The variables $x_i \notin \mathrm{mult}(X^\alpha, \mathcal{F})$ are called non-multiplicative variables for X^α in $\mathcal{F}$.

Definition 2.3.2. *Let X^α be a monomial of the finite set $\mathcal{F} \subset \mathcal{M}(X)$. We call class of X^α in $\mathcal{F}$, noted by $\mathcal{C}_{\alpha,\mathcal{F}}$, the set*

$$C_{\alpha,\mathcal{F}} = \{X^{\alpha+\beta} \mid \text{ Each variable in } X^\beta \text{ belongs to } \mathrm{mult}(X^\alpha, \mathcal{F})\}.$$

Remark 2.3.3. *(Janet,* [13]*, pp. 76-77) Classes corresponding to different monomials are disjoint.*

Definition 2.3.4. ([13], p. 79) *Let $\mathcal{F}$ be a finite subset of $\mathcal{M}(X)$ and denote J the Janet module generated by $\mathcal{F}$. The set $\mathcal{F}$ is said to be complete if the following holds: For all $X^\alpha \in \mathcal{F}$ and for all $x_i \notin \mathrm{mult}(X^\alpha, \mathcal{F})$ there exists $X^\beta \in \mathcal{F}$ such that $x_i X^\alpha \in \mathcal{C}_{\beta,\mathcal{F}}$.*

Let $\mathcal{F}$ be a complete subset of $\mathcal{M}(X)$. Then for each $X^\alpha \in \mathcal{F}$ and for each $x_i \notin \mathrm{mult}(X^\alpha, \mathcal{F})$ the only $X^\beta \in \mathcal{F}$ such that $x_i X^\alpha \in \mathcal{C}_{\beta,\mathcal{F}}$ verifies that $(\alpha_n, \ldots, \alpha_1)$ is less than $(\beta_n, \ldots, \beta_1)$ w.r.t the lexicographical order (see [13], p. 85).

3. Completely integrable systems. Janet bases

Janet considers in [13] (and [14], p. 33) the degree lexicographical order (denoted by $<_{\deg}$ or $\prec$ to short):

$$\alpha <_{\deg} \beta \Longleftrightarrow \begin{cases} |\alpha| < |\beta| \\ \quad \text{or} \\ |\alpha| = |\beta| \quad \text{and} \quad (\alpha_n, \cdots, \alpha_1) <_{\text{lex}} (\beta_n, \cdots, \beta_1), \end{cases}$$

where $<_{\text{lex}}$ is the lexicographical order.

Let $\mathbf{k}$ be a field. We denote by $\mathbf{k}(X)$ (resp. $\mathbf{k}((X))$) the quotient field of the polynomial ring $\mathbf{k}[X]$ (resp. of the formal power series ring $\mathbf{k}[[X]] = \mathbf{k}[[x_1, \ldots, x_n]]$). In this section we will consider the rings of linear differential operators $\mathbf{k}[\partial] = \mathbf{k}[\partial_1, \ldots, \partial_n], Q_n(\mathbf{k}) = \mathbf{k}(X)[\partial_1, \ldots, \partial_n]$ and $\widehat{Q_n}(\mathbf{k}) = \mathbf{k}((X))[\partial_1, \ldots, \partial_n]$. We denote by $\mathcal{R}$ any of these three rings and by $\mathcal{A}$ any of the corresponding fields $\mathbf{k}, \mathbf{k}(X), \mathbf{k}((X))$. Let $\mathcal{N}$ be a left $\mathcal{R}$-module.

Consider a system of (not necessarily homogeneous) linear differential equations:

$$S: \ P_1(u) = f_1, \ldots, P_r(u) = f_r$$

where $P_i \in \mathcal{R}$, $f_i \in \mathcal{N}$ and the unknown u belonging to $\mathcal{N}$. Rewrite the equation $P_j(u) = f_j$ as

$$a_{\alpha(j)}\partial^{\alpha(j)}(u) = \textstyle\sum_{\beta \prec \alpha(j)} a_\beta \partial^\beta(u) + f_j$$

with $a_{\alpha(j)}, a_\beta \in \mathcal{A}$. The element $a_{\alpha(j)}\partial^{\alpha(j)}(u)$ (resp. $\sum_{\beta \prec \alpha(j)} a_\beta \partial^\beta(u) + f_j$) is called the first (resp. second) member of this equation. We will identify $\partial^\alpha(u)$ with ∂^α and with α. When $f_j = 0$ and if no confusion is possible we will identify the equation $P_j(u) = 0$ with the linear differential operator P_j.

Definition 3.1.5. ([13], p. 105) *Let S be a system as above. We say that S is in canonical form (with respect to $\prec$) if the following conditions hold:*
1) $a_{\alpha(j)} = 1$ for all j.
2) The first members of any two equations are distinct.

Definition 3.1.6. ([13], p. 106) *Given a system S in canonical form and $\mathcal{F}$ the set of its first members, we call principal derivative (with respect to S) each monomial ∂^α in the Janet module generated by $\mathcal{F}$. We call parametric derivatives the remaining ones.*

Definition 3.1.7. *Let $E \equiv a_\alpha \partial^\alpha(u) = \sum_{\beta \prec \alpha} a_\beta \partial^\beta(u) + f$ be a linear differential equation with $a_\gamma \in \mathcal{A}$, $f \in \mathcal{N}$ and $a_\alpha \neq 0$. We call support of E the set* $\operatorname{supp}(E) = \{\gamma \in \mathbf{N}^n \mid a_\gamma \neq 0\}$. *We call α the privileged exponent of E (with respect to $\prec$) and we denote it by $\exp_\prec(E)$ (or $\exp(E)$ to short, if no confusion is possible).*

Definition 3.1.8. *Let S be a system as above and denote by $\mathcal{F}$ the set of the first members of S. Let E be an element of S. We call multiplicative (resp. non-multiplicative) variable of E (in S) any of the multiplicative (resp. non-multiplicative) variables of the first member of E (in $\mathcal{F}$). The class of E will be the class of its first member in $\mathcal{F}$ (see 2.3).*

Definition 3.1.9. *The system S is complete if $\mathcal{F}$ is complete (see 2.3.4).*

Let $S = \{E_1, \ldots, E_r\}$ be a complete system of linear homogeneous partial differential equations and suppose the E_i are in canonical form. Let us reproduce the definition of Janet ([13], p. 107): *Si, par dérivations et combinaisons, on ne peut tirer de S aucune relation entre les seules dérivées paramétriques (et les variables indépendantes), on dira que le système est* complètement intégrable.

Denote by I the left ideal (in $\mathcal{R}$) generated by S and write $\Delta(S) = \cup_{j=1}^r (\exp(E_j) + \mathbf{N}^n)$.

Definition 3.1.10. ([13], p. 107) *The complete system S is said to be completely integrable if the only element in I with support in $\mathbf{N}^n \setminus \Delta(S)$ is the zero element.*

Definition 3.1.11. *Let I be a left ideal of $\mathcal{R}$ generated by a finite homogeneous system S. The system S is called a Janet basis (of I) if S is completely integrable.*

4. Janet bases and Gröbner bases

4.1. Homogeneous systems. The theory of Gröbner bases developed by Buchberger [3] for commutative polynomial rings has been generalized to ideals in rings of differential operators and in particular to ideals in $\mathcal{R}$ (see [6, 7], [15], [18], [12], [29]). If I is a left ideal of $\mathcal{R}$ we denote by $\mathrm{Exp}(I)$ the set of privileged exponents $\exp(P)$ for P in I (see 3.1.7). A finite subset $\{P_1, \dots, P_r\} \subset I$ is said to be a Gröbner basis of I if $\mathrm{Exp}(I) = \cup_{j=1}^{r}(\exp(P_j) + \mathbf{N}^n)$.

Given $\underline{\alpha} = (\alpha^1, \dots, \alpha^r) \in (\mathbf{N}^n)^r$ we define the partition $\{\Delta_1, \dots, \Delta_r, \overline{\Delta}\}$ of $\mathbf{N}^n$ associated to $\underline{\alpha}$ as follows:

$$\text{For } i = 1, \dots, r;\ \ \Delta_i = (\alpha^i + \mathbf{N}^n) \setminus (\cup_{j=1}^{i-1}\Delta_j);\ \ \overline{\Delta} = \mathbf{N}^n \setminus (\cup_{i=1}^{r}\Delta_i).$$

If $\underline{E} = (E_1, \cdots, E_r) \in \mathcal{R}^r$ we call partition associated to $\underline{E}$ the partition associated to $(\exp(E_1), \dots, \exp(E_r))$.

Theorem 4.1.1. *(Division theorem in $\mathcal{R}$). Consider $(E_1, \cdots, E_r) \in \mathcal{R}^r$ with $E_i \neq 0$, $i = 1, \cdots, r$. Let $\left\{\Delta_1, \cdots, \Delta_r, \overline{\Delta}\right\}$ the associated partition of $\mathbf{N}^n$. Then, for all $E \in \mathcal{R}$, there exists a unique $(Q_1, \cdots, Q_r, R) \in \mathcal{R}^{r+1}$ such that:*

1. $E = \sum_{i=1}^{r} Q_i E_i + R$.
2. *If $R \neq 0$, each monomial of $\mathcal{R}$ (in the variables $\partial_1, \cdots, \partial_n$) lies in $\overline{\Delta}$.*
3. *If $Q_i \neq 0$, each monomial $c\partial^\alpha$ of Q_i (with $c \in \mathcal{A}$), satisfy $\alpha + exp(E_i) \subseteq \Delta_i$.*

In fact, this division theorem (and its proof) is explicit, although with a different statement, in Janet's work ([13], pp. 100 and 106) when the set $\{E_1, \dots, E_r\}$ is complete and in canonical form.

Proof. The proof is analogous to the commutative polynomial ring case (see for example ([1], p.28) because the coefficients of the differential operators belong to the field $\mathcal{A}$ and because the Leibnitz's rule implies that for all $a \in \mathcal{A}$ and $\alpha \in \mathbf{N}^n$, $\partial^\alpha a - a\partial^\alpha$ is a differential operator of degree less or equal than $|\alpha| - 1 = \alpha_1 + \cdots + \alpha_n - 1$. □

Theorem 4.1.2. *Let I be a left ideal of $\mathcal{R}$ and $\mathcal{B} = \{E_1, \dots, E_r\} \subset I$. If $\mathcal{B}$ is a Janet basis of I then $\mathcal{B}$ is a Gröbner basis of I, with respect to the monomial ordering $\prec$ on $\mathbf{N}^n$.*

Proof. Denote $\Delta = \Delta(\mathcal{B}) = \cup_{j=1}^{r}(\exp(E_j) + \mathbf{N}^n)$. Let P be in I and suppose $\exp(P) \notin \Delta$. By division theorem in $\mathcal{R}$ (see above) there exists $R \in \mathcal{R}$ with $\mathrm{supp}(R) \subset \mathbf{N}^n \setminus \Delta$, such that $P - R \in I$ and $\exp(P) = \exp(R)$. So $R \neq 0$. But this is impossible by the hypothesis (see definitions 3.1.10, 3.1.11). □

We say that a differential operator $P \in \mathcal{R}$ is monic if the coefficient of its privileged monomial is 1.

Proposition 4.1.3. *Let $\mathcal{B} = \{E_1, \dots, E_r\}$ be a Gröbner basis of a left ideal I of $\mathcal{R}$. Suppose $\exp(E_i) \neq \exp(E_j)$, for $i \neq j$, E_i is monic for all i and $\mathcal{B}$ is complete. Then $\mathcal{B}$ is a Janet basis of I.*

Proof. Let R be a non zero element of I with support contained in $\mathbf{N}^n \setminus \Delta(\mathcal{B})$. Then $\exp(R) \in \mathrm{Exp}(I) \bigcap (\mathbf{N}^n \setminus \Delta(\mathcal{B}))$. But $\mathrm{Exp}(I) = \Delta(\mathcal{B})$. □

Theorem 4.1.4. *(Criterion for complete integrability). Let $S = \{E_1, E_2, \cdots, E_r\}$ be a subset of monic elements in $\mathcal{R}$. Suppose that for all i and for all non-multiplicative variable ∂_k for E_i (in S) we have $\partial_k E_i = \sum_{j=1}^{r} A_{ki}^{(j)} E_j$ such that the only variables in each monomial (in the variables $\partial_1, \cdots, \partial_n$) of $A_{ki}^{(j)}$ are multiplicative variables for E_j, $\forall j = 1, 2, \cdots, r$. Then we have:*

1. *For all $H \in \mathcal{R}$,*

$$H \in \langle E_1, E_2, \cdots, E_r \rangle \Longleftrightarrow H = \sum_{i=1}^{r} Q_i E_i$$

   where the only variables of each monomial in Q_i are multiplicative variables for E_i in S. Here $\langle E_1, E_2, \cdots, E_r \rangle$ is the left ideal (of $\mathcal{R}$) generated by the E_i.
2. *S is completely integrable.*

Proof. We can suppose $\exp(E_r) \prec \exp(E_{r-1}) \prec \cdots \prec \exp(E_1)$. The hypothesis implies that S is complete, $\exp(\partial_k E_i) = \exp(A_{ki}^{(c(k,i))} E_{c(k,i)})$ for an unique integer $c(k,i) < i$ (see 2.3) and $\exp(A_{ki}^{(j)} E_j) \prec \exp(\partial_k E_i)$ for $j \neq c(k,i)$.

If $H \in \langle E_1, E_2, \cdots, E_r \rangle$ then we have $H = \sum_{i=1}^{r} G_i E_i$. Each G_i, $i = 1, \cdots, r$ can be writed as $G_i = G_i^{(1)} + H_i$ where $G_i^{(1)}$ is the sum of the monomials of G_i with only multiplicative variables for E_i in S. In particular $H_1 = 0$.

We have

$$H = \sum_{i=1}^{r} G_i E_i = \sum_{i=1}^{r} G_i^{(1)} E_i + \sum_{i=2}^{r} H_i E_i.$$

Let denote $\delta = (\delta_1, \cdots, \delta_n) = \max\ \{\exp(H_i E_i),\ i = 1, \cdots, r\}$ and $i_0 = \max\{i \mid \exp(H_i E_i) = \delta\}$.

We call (δ, i_0) the characteristic exponent of $\sum_{j=1}^{r} H_j E_j$.

We will consider on $\mathbf{N}^n \times \{1, \cdots, r\}$ the well ordering defined as follows:

$$(\delta, i_0) \lhd (\delta', i'_0) \Longleftrightarrow \begin{cases} \delta \prec \delta' \\ \quad \text{or} \\ \delta = \delta' \quad \text{and} \quad i_0 < i'_0. \end{cases}$$

Then we can write

$$H_{i_0} E_{i_0} = a \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} E_{i_0} + \widehat{H_{i_0}} E_{i_0}$$

where $a \in \mathcal{A}$, $\exp(a \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} E_{i_0}) = \delta$ and $\exp(\widehat{H_{i_0}} E_{i_0}) \prec \delta$.

Suppose ∂_k is a non-multiplicative variable for E_{i_0}, then by hypothesis we can write

$$H_{i_0} E_{i_0} = a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} (\partial_k E_{i_0}) + \widehat{H_{i_0}} E_{i_0} =$$

$$a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} \left(\sum_{j=1}^{r} A_{k i_0}^{(j)} E_j \right) + \widehat{H_{i_0}} E_{i_0} =$$

$$= \sum_{j=1}^{r} a \partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A_{k i_0}^{(j)} E_j + \widehat{H_{i_0}} E_{i_0}$$

where the only variables in each monomial of $A^{(j)}_{ki_0}$ are multiplicative variables for E_j. Then rewrite

$$\sum_{i=2}^{r} H_i E_i = H_{i_0} E_{i_0} + \sum_{j \neq i_0} H_j E_j =$$

$$\sum_{j=1}^{r} a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j + \widehat{H_{i_0}} E_{i_0} + \sum_{j \neq i_0} H_j E_j = \sum_{j=1}^{r} H'_j E_j$$

where

$$\begin{cases} H'_j = a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} + H_j \qquad \text{for} \quad j \neq i_0 \\ \\ H'_{i_0} = a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(i_0)}_{ki_0} + \widehat{H_{i_0}}. \end{cases}$$

Now we will compute the characteristic exponent of this new expression:

1. For $i_0 + 1 \leq j \leq r$ we have $\exp(H'_j E_j) = \exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j + H_j E_j) \preceq \max\{\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j), \exp(H_j E_j)\}$. We have first $\exp(H_j E_j) \prec \delta$, because the definition of i_0, and then

$$\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j) = (\alpha_1, \cdots, \alpha_k - 1, \cdots, \alpha_n) + \exp(A^{(j)}_{ki_0} E_j) \prec$$
$$(\alpha_1, \cdots, \alpha_k - 1, \cdots, \alpha_n) + \exp(\partial_k E_{i_0}) = \delta,$$

   So, $\exp(H'_j E_j) \prec \delta$ for $i_0 + 1 \leq j \leq r$.
2. $\exp(H'_{i_0} E_{i_0}) = \exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(i_0)}_{ki_0} E_{i_0} + \widehat{H_{i_0}} E_{i_0}) \preceq$

$$\max\{\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(i_0)}_{ki_0} E_{i_0}), \exp(\widehat{H_{i_0}} E_{i_0})\}$$

   and then $\exp(H'_{i_0} E_{i_0}) \prec \delta$.
3. For $1 \leq j \leq i_0 - 1$ we have

$$\exp(H'_j E_j) = \exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j + H_j E_j) \prec$$
$$\max\{\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j), \exp(H_j E_j)\}.$$

   The choice of j implies that $\exp(H_j E_j) \preceq \delta$ and, on the other hand, we have

$$\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} A^{(j)}_{ki_0} E_j) = (\alpha_1, \cdots, \alpha_k - 1, \cdots, \alpha_n) + \exp(A^{(j)}_{ki_0} E_j) \preceq$$
$$\exp(a\partial_1^{\alpha_1} \cdots \partial_k^{\alpha_k - 1} \cdots \partial_n^{\alpha_n} \partial_k E_{i_0}) \preceq \delta.$$

   So, the characteristic exponent (δ', i'_0) of $\sum_j H'_j E_j$ is less than (δ, i_0) w.r.t the well ordering $\lhd$, which implies the assertions of the theorem.

$\square$

Remark 4.1.5. *As a consequence of this theorem, Janet develops a finite procedure constructing a completely integrable system (i.e. a Janet basis) of a left ideal I of $\mathcal{R}$, starting from an arbitrary system of generators of I. This algorithm should be compared to Buchberger's algorithm computing Gröbner bases. Janet's procedure is as follows: a) We can suppose the starting system $S_1 = \{E_1, \ldots, E_r\}$ complete and in canonical form (see 3.1.9, 3.1.5). b) For each $i = 1, \ldots, r$ and each k such that ∂_k is non-multiplicative for E_i, write (see 4.1.1) $\partial_k E_i = \sum_{j=1}^{r} A^{(j)}_{ki} E_j + R_{ik}$ where*

1. *Each monomial in $A^{(j)}_{ki}$ (in $\partial_1, \ldots, \partial_n$) is formed only by multiplicative variables for E_j in S_1.*
2. *The support of R_{ki} is contained in $\mathbf{N}^n \setminus \Delta(S_1)$.*

c) If all the R_{ki} are zero, then S_1 is completely integrable (see 4.1.4). d) If there exists $R_{ki} \neq 0$ then we consider the new system $S_2 = S_1 \cup \{R_{ki}\}$ and we restart.

This procedure is finite. Indeed, let S_i, $i = 1, 2, \ldots$ be the sequence of systems obtained applying Janet's procedure. Write $F_i = \{\exp(E) \mid E \in S_i\} \subset \mathbf{N}^n$. By 2.1.2 this sequence is stationary and the procedure is finite.

4.2. Non-homogeneous systems. In this section we will explain how to extend the results of 3 and 4 to system of linear non-homogeneous differential equations.

Let S be a system of linear, non necessarily homogeneous, differential equations

$$P_1(u) = f_1, \cdots, P_r(u) = f_r$$

where $P_i \in \mathcal{R}$, $f_i \in \mathcal{N}$ and the unknown u belonging to $\mathcal{N}$. We denote by S^h the homogeneous system $P_1(u) = \cdots = P_r(u) = 0$ associated to S.

We will denote by E_i the equation $P_i(u) = f_i$ (or $P_i(u) - f_i = 0$).

We identify the equation $P_i(u) = f_i$ (i.e. the equation E_i) with the couple $(P_i, f_i) \in \mathcal{R} \oplus \mathcal{N}$ and we consider the $\mathcal{R}$-sub-module M of $\mathcal{R} \oplus \mathcal{N}$ generated by $\{(P_1, f_1), \cdots, (P_r, f_r)\}$.

Definition 4.2.1. *Let $S = \{E_1, \cdots, E_r\} = \{(P_1, f_1), \cdots, (P_r, f_r)\}$ be a complete system in canonical form. Let M be the $\mathcal{R}$-sub-module of $\mathcal{R} \oplus \mathcal{N}$ generated by S. The system S is said to be completely integrable if the following holds:*
1) If $(0, f) \in M$ then $f = 0$.
2) If $(P, f) \in M$ and $P \neq 0$ then the support of P is not contained in $\mathbf{N}^n \setminus \Delta(S^h)$.

Definition 4.2.2. *Let M be the $\mathcal{R}$-sub-module of $\mathcal{R} \oplus \mathcal{N}$ generated by $S = \{(P_1, f_1), \cdots, (P_r, f_r)\}$. We call S a Janet basis of M if S is completely integrable.*

Denote $\mathcal{E}$ the (left) $\mathcal{R}$-module $\mathcal{R} \oplus \mathcal{N}$ and $\pi_1 : \mathcal{E} \longrightarrow \mathcal{R}$ the canonical projection.

As in $\mathcal{R}$ we have in $\mathcal{E}$ the notions of privileged exponent and Gröbner basis and we have a division theorem in $\mathcal{E}$.

We still denote $\exp : \mathcal{E} \setminus (\{0\} \oplus \mathcal{N}) \longrightarrow \mathbf{N}^n$ the map $\exp_{\prec}(P, f) = \exp(P)$.

Theorem 4.2.3. *(Division theorem in $\mathcal{E}$). Consider $(E_1, \cdots, E_r) \in \mathcal{E}^r$ with $E_i = (P_i, f_i)$ and $P_i \neq 0$, $i = 1, \cdots, r$. Let $\left\{\Delta_1, \cdots, \Delta_r, \overline{\Delta}\right\}$ the associated partition of $\mathbf{N}^n$. Then, for all $E = (P, f) \in \mathcal{E}$, there exists a unique $(Q_1, \cdots, Q_r, (R, g)) \in \mathcal{R}^r \times \mathcal{E}$ such that:*

1. *$E = \sum_{i=1}^r Q_i E_i + (R, g)$.*
2. *If $R \neq 0$, each monomial of R (in the variables $\partial_1, \cdots, \partial_n$) lies in $\overline{\Delta}$.*
3. *If $Q_i \neq 0$, each monomial $c\partial^\alpha$ of Q_i (with $c \in \mathcal{A}$), satisfy $\alpha + \exp(E_i) \subseteq \Delta_i$.*

Proof. Analogous to the proof of 4.1.1. We first write $P = \sum_{i=1}^r Q_i P_i + R$ and then $g = f - \sum_{i=1}^r Q_i(f_i)$. □

Definition 4.2.4. *Let M be a $\mathcal{R}$-sub-module of $\mathcal{E}$. A finite subset $\{(P_1, f_1), \cdots, (P_m, f_m)\}$ of M is said to be a Gröbner basis of M, with respect to $\prec$, if the following two conditions hold:*

1. *$\{P_1, \cdots, P_m\}$ is a Gröbner basis of $\pi_1(M)$ with respect to $\prec$.*
2. *For all $g \in \mathcal{N}$, if $(0, g) \in M$ then $g = 0$.*

Theorem 4.2.5. *Let M be a $\mathcal{R}$-sub-module of $\mathcal{E} = \mathcal{R} \oplus \mathcal{N}$ and suppose $\mathcal{B} = \{(P_1, f_1), \cdots, (P_r, f_r)\} \subseteq M$ is a Janet basis of M. Then $\mathcal{B}$ Gröbner basis of M, with respect to $\prec$.*

Proof. Analogous to the proof of 4.1.2, using the division theorem 4.2.3. □

Proposition 4.2.6. *Let $\mathcal{B} = \{E_1 = (P_1, f_1), \ldots, E_r = (P_r, f_r)\}$ be a Gröbner basis of a left $\mathcal{R}$-sub-module M of $\mathcal{R} \oplus \mathcal{N}$. Suppose* $\exp(P_i) \neq \exp(P_j)$ *for $i \neq j$, E_i is monic for all i and $\mathcal{B}$ is complete. Then $\mathcal{B}$ is a Janet basis of M.*

Proof. Suppose $(P, f) \in M$ and $\text{supp}(P) \subset \mathbf{N}^n \setminus \Delta(S^h)$. The family $\{P_1, \ldots, P_r\}$ is a Gröbner basis of the ideal $\pi_1(M)$ and then $\Delta(S^h) = \text{Exp}(\pi_1(M))$. So, $\exp(P) \in \Delta(S^h)$ and then $P = 0$. □

Theorem 4.2.7. *(Criterion for complete integrability). Let $S = \{E_1, E_2, \cdots, E_r\}$ be a subset of monic elements in $\mathcal{E}$. Suppose that for all i and for all non-multiplicative variable ∂_k for E_i (in S) we have $\partial_k E_i = \sum_{j=1}^r A_{ki}^{(j)} E_j$ where the only variables in each monomial (in the variables $\partial_1, \partial_2, \cdots, \partial_n$) of $A_{ki}^{(j)}$ are multiplicative variables for E_j, $\forall j = 1, 2, \cdots, r$. Then we have:*

1. *For all $H \in \mathcal{E}$,*
$$H \in \langle E_1, E_2, \cdots, E_r \rangle \Longleftrightarrow H = \sum_{i=1}^r Q_i E_i$$
where the only variables of each monomial in Q_i are multiplicative variables for E_i in S. Here $\langle E_1, E_2, \cdots, E_r \rangle$ is the left $\mathcal{R}$-module generated by the E_i.
2. *S is completely integrable.*

Proof. Analogous to the proof of 4.1.4. □

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