PREPUBLICACIONES DEL DEPARTAMENTO DE ÁLGEBRA

DE LA UNIVERSIDAD DE SEVILLA

On the Graver Basis of Semigroup Ideals

P. Pisón-Casares, A. Vigneron-Tenorio

Prepublicación nº 10 (Mayo-2001)

Departamento de Álgebra. Universidad de Sevilla

On the Graver Basis of Semigroup Ideals *

P. Pisón-Casares
Dpto. de Álgebra. Universidad de Sevilla
ppison@cica.es
A. Vigneron-Tenorio
Dpto. de Matemáticas. Universidad de Cádiz
alberto.vigneron@uca.es

Abstract

The Graver basis of a semigroup ideal is computed from a minimal generating set for its Lawrence lifting. A combinatorial characterization of the minimal degrees of a Lawrence ideal is given as well as a degree bound for its minimal first syzygies.

Keywords: Semigroup ideal, Gröbner basis, Graver basis, simplicial complex, Lawrence lifting semigroup

Introduction

The Graver basis of a toric ideal can be computed by using any reduced Gröbner basis of its Lawrence lifting [6]. Concretely, it is enough to substitute some variables by 1 in the elements of this Gröbner basis. In section 1, we show how this method can be used, taking a more general class of ideals. This generalization consists of admiting torsion in the associated semigroup.

The Graver basis of a Lawrence ideal is a minimal system of generators, hence this minimal system is unique except scalar multiples. In this paper, we characterize the degrees of this system by means of a property of symmetry for the connected components of some simplicial complexes (Proposition 7). The property is described in the context of integer programming (Definition 6). The result provides an algorithm for computing the minimal generating set for a Lawrence ideal (Algorithm 3). Our characterization is obtained from the general for the minimal degrees of a semigroup ideal that appeared in [3, Therorem 1]. Concretely, we use the effective version of this result, [1, Theorem 3.11].

On the other hand, using the techniques in [5], we study the first syzygies of Lawrence ideals. As an application, we give an explicit degree bound for the minimal first syzygies of these ideals (Theorem 13).

*Supported by MCyT Spain, BFM2000-1523

1 Graver Basis and the Lawrence Lifting of Semigroups

Let

$$S \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/a_h\mathbf{Z}$$

be a finitely generated semigroup with zero element, and $\{n_1, \ldots, n_r\} \subset S$ a set of generators for S.

Fixing k as a commutative field, one can consider the semigroup $k-$algebra associated to S, $k[S] = \oplus_{m \in S} \chi^m$ and $k[X] = k[x_1, \ldots, x_r]$, the polynomial ring in r indeterminates where the $S-$degree of x_i is equal to n_i. We denote by X^α, where $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbf{N}^r$, the monomial $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$. Let $\gg$ be the partial natural order on $\mathbf{N}^r$.

The $k-$algebra epimorphism,

$$\varphi : k[X] \longrightarrow k[S],$$

defined by $\varphi(x_i) = \chi^{n_i}$, is an $S-$graded homomorphism of zero degree, and $I = ker(\varphi)$ is a homogeneous ideal, which we shall call *the ideal of* S.

It is well-known ([4]) that

$$\mathcal{B} = \left\{ X^\alpha - X^\beta | \sum_{i=1}^{r} \alpha_i n_i = \sum_{i=1}^{r} \beta_i n_i,\ \alpha_i, \beta_i \geq 0 \right\}$$

is a set of generators for I. In this paper we will consider systems of pure binomial generators, i.e. subsets of $\mathcal{B}$ only.

In a semigroup ideal there are some special binomials: $X^\alpha - X^\beta$ is called *primitive* if there exists no binomial $X^{\alpha'} - X^{\beta'}$ in the ideal such that $\alpha \gg \alpha'$ and $\beta \gg \beta'$. Note that $X^{\alpha'}$ divides X^α iff $\alpha \gg \alpha'$.

Now, we can consider

$$Gr_I = \{X^\alpha - X^\beta \in I |\, X^\alpha - X^\beta \text{ is primitive}\}$$

This set is a finite system of generators for I and it is called *Graver basis* for I (see [6]).

One way to obtain the Graver basis is by using diophantine equations in congruence. If one take the $((n+h) \times r)-$matrix

$$\mathcal{A} = (n_1 | n_2 | \cdots | n_r) \in \mathcal{M}_{(h+n) \times r}(\mathbf{Z}),$$

considering $n_1, \ldots, n_r \in \mathbf{Z}^{h+n}$, one can prove $X^\alpha - X^\beta \in Gr_I$ iff (α, β) is a $\gg -$minimal $\mathbf{N}-$solution, with $\alpha \neq \beta$, of the system

$$(\mathcal{A} | -\mathcal{A})Y = 0 \; mod\, a,$$

where the last h rows of this system are in congruence.

However, it is more useful for explicit computations to use the generalization of the techniques in [6]. Concretly, to compute Gr_I we construct a new semigroup

$$S' = <(n_1, e_1), \dots, (n_r, e_r), (0, e_1), \dots, (0, e_r)> \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_h \oplus \mathbf{Z}^r$$

with $\{e_1, \dots, e_r\}$ the standard coordinate vectors in $\mathbf{Q}^r$, which one calls *Lawrence Lifting of S*.

Generally, the semigroups like S' are called *Lawrence semigroups.* In this section, we denote $k_{S'}[X] = k[x_1, \dots, x_r, x_{r+1}, \dots, x_{2r}]$ and $I_{S'}$ the ideal of S'.

The relation between Gr_I and $Gr_{I_{S'}}$ is the following:

Proposition 1. *Let*

$$S \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/a_h\mathbf{Z}$$

be a finitely generated semigroup with zero element and let S' be its Lawrence lifting. Then

$$Gr_I = \{f(x_1, \dots, x_r, 1, \dots, 1) \mid f \in Gr_{I_{S'}}\}.$$

Proof. It is a generalization for non torsion free semigroups of [6, Algorithm 7.2] □

The new semigroup S', like all Lawrence semigroups, satisfies the following theorem:

Theorem 2. *For a Lawrence semigroup, S', the following sets coincide:*

1. *any minimal system of generators for $I_{S'}$ (except scalar multiples)*
2. *the Graver basis for $I_{S'}$,*
3. *the universal Gröbner basis for $I_{S'}$,*
4. *any reduced Gröbner basis for $I_{S'}$.*

Proof. It is a generalization for non torsion free semigroups of [6, Theorem 7.1]. □

Corollary 3. *For Lawrence ideal, except scalar multiples:*

- *Only a minimal system of generators exists.*
- *Only a reduced Gröbner basis exists.*

Using this Theorem and Proposition 1, one has an algorithm based in [6, Algorithm 7.2] computing the Graver basis for I_S.

Algorithm 1. *Graver Basis*
In: *Lawrence lifting of semigroup S, S'*
Out: *Graver basis of I.*

1. *Compute the minimal set of generators of* $I_{S'} \in k_{S'}[X]$.
2. *Take* $H = \{f(x_1, \ldots, x_r, 1, \ldots, 1) \;\mid\; f \in Gr_{I_{S'}}\}$.
3. $Gr_I = H$.

There are different algorithms to compute $I_{S'}$ using Gröbner basis (see [7]).

2 Combinatoric Results over Lawrence Ideals

For the following sections, we fix the Lawrence semigroup

$$S = < n'_1, \ldots, n'_r, n'_{r+1}, \ldots, n'_{2r} > \subset \mathbf{Z}^n \oplus \mathbf{Z}/a_1 \oplus \cdots \oplus \mathbf{Z}/a_h \oplus \mathbf{Z}^r$$

where $n'_i = (n_i, e_i)$ for all $i = 1, \ldots, r$, and $n'_i = (0, e_{i-r})$, $\forall i = r+1, \ldots, 2r$. Thus, $\Lambda = \{1, \ldots, r, r+1, \ldots, 2r\}$. Note that $S \cap (-S) = \{0\}$. One can consider the simplicial complex

$$\Delta_m = \{F \subset \Lambda |\, m - n'_F \in S\},$$

where $n'_F = \sum_{i\in F} n'_i$. Notice that, if F is a maximal face in Δ_m, then there is a monomial of degree m with support F.

We denote by $sym()$ to the function

$$sym(i) = \begin{cases} i + r & , \quad i \le r \\ i - r & , \quad i > r \end{cases} ,i \in \Lambda$$

Let $A \subset \Lambda$, set $sym(A) := \{sym(i)|\, i \in A\}$. In the following lemma we prove an important property of the simplicial complexes Δ_m associated to a Lawrence semigroup.

Lemma 4. *If* $\{i\} \in \Delta_m$ *and* $m = \sum_{j\in G} \gamma_j n'_j$, *where* $i \notin G \subset \Lambda$, *then* $sym(i) \in G$.

Proof. $\{i\} \in \Delta_m$ implies that $m - n'_i \in S$. Thus, if $n'_i = (*, e_l)$ then $(h+n+l)-$th coordinate of m is non null. Since $i \notin G$ and $m = \sum_{j\in G} \gamma_j n'_j$, it is clear that $sym(i) \in G$. □

Now we can write the following Proposition:

Proposition 5. *Let* Δ_m *be non connected. Then:*

1. $\Delta_m = C \sqcup sym(C)$, *where* C *and* $sym(C)$ *are the only two connected components of* Δ_m.
2. $1 \le \sharp C \le r$.
3. C *and* $sym(C)$ *are full subcomplexes.*

Proof. 1. Let A, B be two different connected components of Δ_m, then $A \cap B = \emptyset$ and $m = \sum_{i \in A} \alpha_i n'_i = \sum_{i \in B} \beta_i n'_i$. Using Lemma 4 one obtains $sym(A) \subset B$ and $sym(B) \subset A$. As $sym()$ is an idempotent function, the equalities are true.

2. We know $C \sqcup sym(C) \subseteq \Lambda$. Then if $\sharp C > r$, $\sharp \Lambda > 2r$, but this is not possible.

3. Suppose C is not a full subcomplex, in that case there exist $A, B \subset C$ maximal faces of C, such that $A \neq B$ and $\exists \alpha_i, \beta_i \in \mathbf{N} \setminus \{0\}$, $m = \sum_{i \in A} \alpha_i n'_i = \sum_{i \in B} \beta_i n'_i$.
Let $i \in A$ and $i \notin B$, then $sym(i) \in B$. We have $i, sym(i) \in C$, but this is not possible.

□

Considering the set $A = \{i_1, \ldots, i_s\} \subset \Lambda$, we denote

$$N(A) = \left\{ (\alpha_{i_1}, \ldots, \alpha_{i_s}) \in (\mathbf{N} \setminus \{0\})^s | \sum_{j=1}^{s} \alpha_{i_j} n'_{i_j} = \sum_{j=1}^{s} \alpha_{i_j} n'_{sym(i_j)} \right\}.$$

Note that

$$N(A) = \left\{ \alpha \in \mathbf{N}^s | \left(\ (n'_{i_1} - n'_{sym(i_1)}) | \cdots | (n'_{i_s} - n'_{sym(i_s)}) \ \right) \alpha = 0, \ \alpha \gg (1, 1, \ldots, 1) \right\}.$$

By Dickson's lemma, the set

$$\mathcal{H}(A) = \{\alpha \in N(A) | \ \alpha \text{ is minimal for } \gg\}$$

is finite.

Definition 6. *Let $A = \{i_1, \ldots, i_s\} \subset \Lambda$ and $m \in S$. We shall say A is $m-$symmetrical if the following conditions are satisfied:*

1. *$\exists (\alpha_{i_1}, \ldots, \alpha_{i_s}) \in \mathcal{H}(A)$ such that $m = \sum_{j=1}^{s} \alpha_{i_j} n'_{i_j}$.*

2. *$(\alpha_{l_1}, \ldots, \alpha_{l_t}) \notin N(A')$, $\forall A' = \{l_1, \ldots, l_t\} \subset A$, $1 \leq t \leq s-1$.*

In this case, we denote $\mathcal{M}_m(A) = x_{i_1}^{\alpha_{i_1}} \cdots x_{i_s}^{\alpha_{i_s}}$

Proposition 7. *Let S be a Lawrence semigroup, and let $m \in S$. The following conditions are equivalent:*

1. *m is a minimal degree.*

2. *$\exists C \subset \Lambda$ $m-$symmetrical.*

Proof. Note that if S is a Lawrence semigroup, our concept of $m-$symmetrical is equivalent to the concept of $m-$chain isolating C from $\Lambda \setminus C$ appears in [1]. The proof of this Proposition is analogous to [1, Theorem 3.11]. □

Corollary 8.

$$\{\mathcal{M}_m(C) - \mathcal{M}_m(sym(C)) |\, m \in S \text{ and } C \text{ is } m-symmetrical\}$$

is a minimal set of generators for I.

Proof. See [1, Theorem 2.5]. □

Corollary 9. *If I is a Lawrence ideal, for any $m \in S$ minimal degree, there exists a unique binomial of degree m in the generating set.*

Let $C \subset \Lambda$, to determine whether there exists $m \in S$ such that C is $m-$symmetrical, we can use the following algorithm.

Algorithm 2. *$m-$symmetrical Algorithm*
In: $C = \{i_1, \ldots, i_s\} \subset \Lambda$.
Out: *Detect whether there exists $m \in S$ such that C is $m-$symmetrical, and, in the case of this being so, all the possible m's, $\mathcal{M}_m(C)$ and $\mathcal{M}_m(sym(C))$ return.*

1. *If $C \cap sym(C) \neq \emptyset$, then there exists no m, STOP.*
2. *Compute $\mathcal{H}(C)$ (see Algorithm in section 1 of [5]).*
3. *Take*

   $$T = \mathcal{H}(C) \backslash \{\alpha \in \mathcal{H}(C) |\, \exists C' = \{l_1, \ldots, l_t\} \subset C \text{ and } (\alpha_{l_1}, \ldots, \alpha_{l_t}) \in N(C'), t < s\}.$$

4. *If $T = \emptyset$, then there exists no m, STOP.*
5. *For each $\alpha \in T$, C is $m-$symmetrical for $m = \alpha_1 n'_{i_1} + \cdots + \alpha_s n'_{i_s}$, $\mathcal{M}_m(C) = x_{i_1}^{\alpha_1} \cdots x_{i_s}^{\alpha_s}$ and $\mathcal{M}_m(sym(C)) = x_{sym(i_1)}^{\alpha_1} \cdots x_{sym(i_s)}^{\alpha_s}$.*

To finish this section, a combinatorial algorithm computing the ideal of a Lawrence semigroup is given.

Algorithm 3. *Minimal Generators*
In: $S := <(n_1, e_1), \ldots, (n_r, e_r), (0, e_1), \ldots, (0, e_r)>$ *Lawrence semigroup.*
Out: *Minimal system of generators for I.*

1. *$G = \emptyset$.*
2. *Let $\mathcal{C}(\Lambda) = \{C \in \mathcal{P}(\Lambda) |\, 1 \leq \sharp C \leq r\}$.*
3. *For each $C \in \mathcal{C}(\Lambda)$ do*
   (a) *Using algorithm 2 the elements $m \in S$ such that C is $m-$symmetrical are computed.*
   (b) *$G = G \cup \{\mathcal{M}_m(C) - \mathcal{M}_m(sym(C)) | C \text{ is } m-symmetrical\}$.*
4. *G is* the *minimal system of generators for I.*

3 Combinatoric Results over First Syzygies of Lawrence Ideals

In this section, we are going to make a combinatorial study of the first syzygies of our ideal I. First of all, we introduce the concept of $F-$cavity (see [5] for details).

Definition 10. *Let $m \in S$ and $F = \{i_1, \dots, i_t\} \subset \Lambda$ such that $\sharp F \geq 3$, and let σ be a polygon whose vertex set is F. We say σ is an $F-$cavity of Δ_m if the following conditions are satisfied:*

1. *$F_j \in \Delta_m$, $\forall j = 1, \dots, t$ where*

$$F_j = \{i_j, i_{j+1}\}, \forall j = 1, \dots, t-1, \text{ and } F_t = \{i_t, i_1\},$$

are the faces of σ.

2. *If $F_j \neq F' \subset F$, $\sharp F' \geq 2$, then $F' \notin \Delta_m$.*

The relation between the $F-$cavities and the degrees of the first syzygies is the following ([5, Lemma 13]):

Lemma 11. *Let $m \in S$ such that $\tilde{H}_1(\Delta_m) \neq 0$. Then, there is σ an $F-$cavity of Δ_m with faces F_i satisfying*

$$c = \sum_{j=1}^{t} \epsilon_j F_j \in \tilde{H}_1(\Delta_m) \setminus \{0\},$$

for some $\epsilon_j = \pm 1$, $\forall j = 1, \dots, t$.

The particular nature of the Lawrence semigroups permit the following result:

Proposition 12. *In the conditions of Lemma 11,*

$$3 \leq \sharp F \leq 5.$$

Moreover, σ has one of the shapes in figure 1.

Proof. Let $\sigma = \{F_1, \dots F_t\}$ be a $F-$cavity of Δ_m as in Definition 10, then there is $\alpha^{(i)} = (\alpha_1^{(i)}, \dots, \alpha_{2r-t+2}^{(i)}) \in \mathbf{N}^{(2r-t+2)t}$ satisfying

$$\begin{cases} m &= n'_{F_1} + \sum_{j \in G_1} \alpha_j^{(1)} n'_j \\ m &= n'_{F_2} + \sum_{j \in G_2} \alpha_j^{(2)} n'_j \\ & \vdots \\ m &= n'_{F_{t-1}} + \sum_{j \in G_{t-1}} \alpha_j^{(t-1)} n'_j \\ m &= n'_{F_t} + \sum_{j \in G_t} \alpha_j^{(t)} n'_j \end{cases}$$

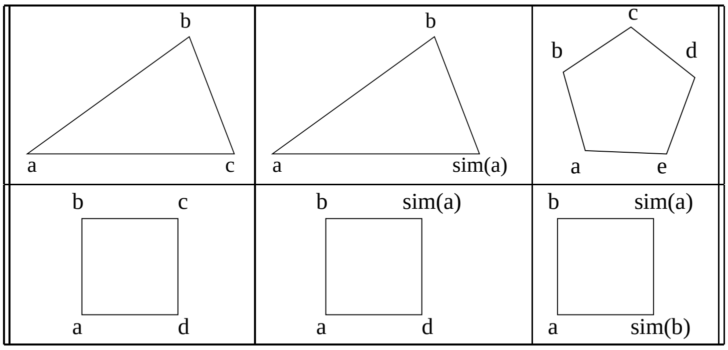

Figure 1: Possible $F-$cavities

where $G_l := (\Lambda \setminus F) \cup F_l$ for $l = 1, \dots, t$. By Lemma 4, $i \in F \setminus F_l$ implies $sym(i) \in G_l$. Therefore, $\cup_{i \in F \setminus F_l} sym(i) \subset G_l$ for all $l = 1, \dots, t$, and in particular, $\cup_{F_j \cap F_l = \emptyset} sym(F_j) \subset G_l$.

First, we are going to prove the $F-$cavities where $\sharp F \geq 5$ do not contain a vertex and its symmetry. Suppose, for example, $i_1, sym(i_1) \in F$. Since $\sharp F \geq 5$, there is an l, such that $i_1, sym(i_1) \notin F_l$. Then, one can write m without using $n'_{i_1}, n'_{sym(i_1)}$. But it is impossible.

Suppose now that $\sharp F > 5$. Notice that the following sets are in Δ_m

$$\underbrace{sym(\cup_{F_j \cap F_1 = \emptyset} F_j), sym(\cup_{F_j \cap F_2 = \emptyset} F_j), \dots, sym(\cup_{F_j \cap F_t = \emptyset} F_j)}_{\text{Full Top (base)}},$$
$$\underbrace{F_1 \cup sym(F_3), F_2 \cup sym(F_4), \dots, F_{t-2} \cup sym(F_t), F_{t-1} \cup sym(F_1), F_t \cup sym(F_2)}_{\text{Full Sides}}$$

Thus, Δ_m contains a prism with an empty top $\{i_1, \dots, i_t\}$ (like Figure 2), full base $\{sym(i_1), \dots, sym(i_t)\}$, and full sides. The topological invariance of the simplicial homology groups yields $c = 0$ as an element in $\tilde{H}_1(\Delta_m)$. This is a contradiction with lemma 11. Therefore, $\sharp F \leq 5$.

We have just seen the unique possibility for an $F-$cavity with $\sharp F = 5$ in Figure 1. If $\sharp F$ is equal to 3 or 4, with similar technical reasonings, one can prove the only possibilities for the $F-$cavities are in that figure.

$\square$

The preceding result let us improve, for Lawrence Semigroups, the bound degree that appears in [5]. Here, $\mathcal{A} = (n'_1 | \dots | n'_{2r})$ and

$$||\mathcal{A}|| = \sup_l \sum_j |a_{lj}|$$

.

Theorem 13. *Let S be a Lawrence semigroup, and let $m \in S$ be a* degree *of an element of a minimal system of homogeneous generators for the first syzygy*

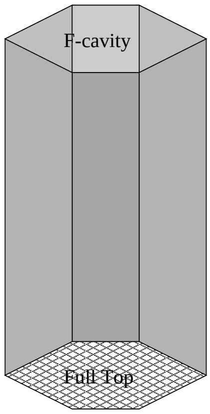

Figure 2: $\sharp F > 5$.

module of $k[S]$ *with* S *a Lawrence semigroup. Then* $\exists \alpha \in \mathbf{N}^{2r}$ *such that* $m = \sum \alpha_i n'_i$ *and* $||\alpha||_1$ *is at most*

$$(1 + 2 \max_{i=1,\dots,h} \{|a_i|\} + 4||\mathcal{A}||)^{4(h+n+r)} + 9.$$

Proof. Analogous to [5, Theorem 23] (see also [2]). □

4 Examples

We are going to illustrate the computation of the minimal systems of generators for a Lawrence ideal and the $F-$cavities associated to its first syzygies.

Let $S \subset \mathbf{Z} \oplus \mathbf{Z}/3 \oplus \mathbf{Z}^3$ be the Lawrence semigroup generated by

$$< (0,2,1,0,0), (2,1,0,1,0), (2,2,0,0,1), (0,0,1,0,0), (0,0,0,1,0), (0,0,0,0,1) > .$$

First of all, we apply Algorithm 3 for any $C \in \mathcal{C}(\Lambda)$.

As an example, we consider $C = \{2, 6\}$ and the system of diophantine equations in congruence

$$(n_2 - n_5 | n_6 - n_3)\alpha = 0 \equiv \left\{ \begin{array}{cccc} \alpha_1 & -\alpha_2 & = & 0 \\ \alpha_1 & -2\alpha_2 & = & 0 \quad mod\ 3 \end{array} \right.$$

Using the algorithms that appear in [5, Section 1], one can compute the set $T = \{(3,3)\}$. This means C is $(6,0,0,3,3)-$symmetrical and then the binomial

$$\mathcal{M}_{(6,0,0,3,3)}(C) - \mathcal{M}_{(6,0,0,3,3)}(sym(C)) = x_2^3 x_6^3 - x_3^3 x_5^3,$$

is in the minimal system of generator for the ideal of S.

Continuing this algorithm with all the elements of $\mathcal{C}(\Lambda)$, we obtain the minimal $S-$degrees m, their $m-$symmetrical components (Figure 3) and their associated binomials.

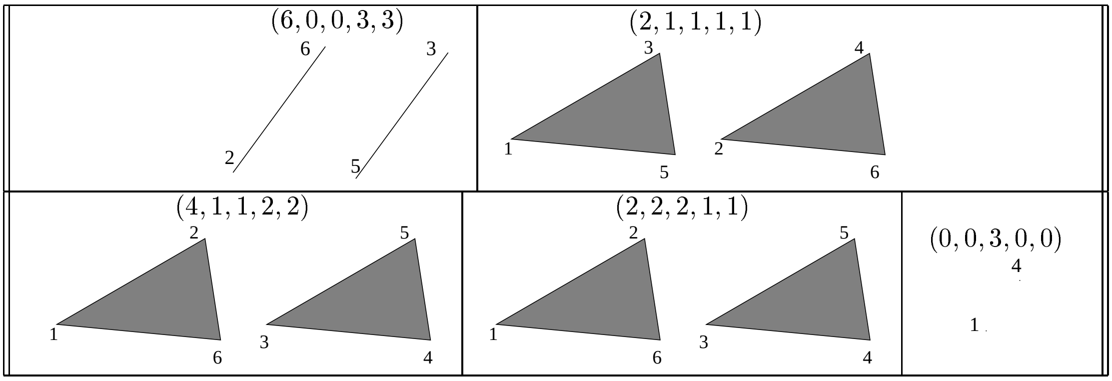

Figure 3: $m-$degrees

The minimal system of generator for I is:

$$
\begin{aligned}
gen_1 &= x_2^3x_6^3 - x_3^3x_5^3 \\
gen_2 &= x_1x_3x_5 - x_2x_4x_6 \\
gen_3 &= x_1x_2^2x_6^2 - x_3^2x_4x_5^2 \\
gen_4 &= x_1^2x_2x_6 - x_3x_4^2x_5 \\
gen_5 &= x_1^3 - x_4^3
\end{aligned}
$$

The minimal system of generators for the first syzygies is

$$
\begin{array}{l}
x_2^2x_6^2gen_2 - x_3x_5gen_3 + x_4gen_1, x_1gen_1 - x_2x_6gen_3 + x_3^2x_5^2gen_2, \\
x_1gen_3 - x_2x_6gen_4 + x_3x_4x_5gen_2, x_1gen_4 - x_2x_6gen_5 + x_4^2gen_2, \\
x_1x_2x_6gen_2 - x_3x_5gen_4 + x_4gen_3, x_1^2gen_2 - x_3x_5gen_5 + x_4gen_4
\end{array}
$$

The σ $F-$cavities associated to their degrees are in Figure 4.

References

[1] E. BRIALES, A. CAMPILLO, C. MARIJUÁN, P. PISÓN, Minimal Systems of Generators for Ideals of Semigroups, J. of Pure and Applied Algebra **127**, (1998) 7-30.

[2] E. BRIALES, P. PISÓN, A. VIGNERON, The Regularity of a Toric Variety, Journal of Algebra **237**, (2001) 165-185.

[3] CAMPILLO, P. PISÓN, L'idéal d'un semi-grupe de type fini, C. R. Acad. Sci. Paris, Série I **316**, (1993) 1303-1306.

[4] J. HERZOG, Generators of Relations of Abelian Semigroups and Semigroups Ring, Manuscripta Math. **3**, (1970) 175-193.

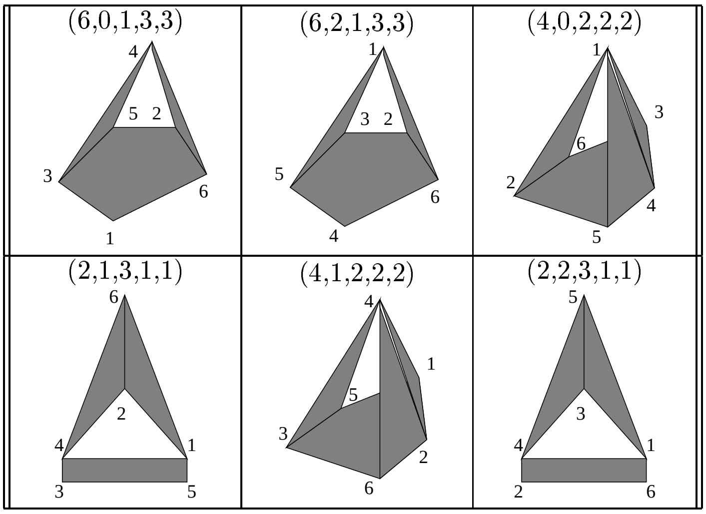

Figure 4: $m-$degrees

[5] P. PISÓN-CASARES, A. VIGNERON-TENORIO, Fisrt Syzigies of Toric Varieties and Diophantine Equations in Congruence, Comm. Alg. **29**(4) (2001).

[6] B. STURMFELS, *Gröbner Bases and Convex Polytopes,* American Mathematical Society, University Lecture Series, **8**, Providence, RI, 1995.

[7] A. VIGNERON-TENORIO, Semigroups Ideals and Linear Diophantine Equations, Linear algebra and its applications **295**, (1990) 133-144.