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Abstract

The short resolution of a lattice ideal is a free resolution over a polynomial ring whose number of variables is the number of extremal rays in the associated cone. A combinatorial description of this resolution is given. In the homogeneous case, the regularity can be computed from this resolution.

Introduction

Let I be a lattice ideal in $k[X_1, \ldots, X_n] = k[\mathbf{X}]$, where k is a (commutative) field. The minimal free resolution of I as $k[\mathbf{X}]$-module has been studied by many authors. Recently combinatorial descriptions of this resolution have been given (see for example [2], [5] and references therein). In this paper we consider the minimal free resolution of I, not over $k[\mathbf{X}]$ but over a polynomial ring over k whose number of variables is the number of extremal rays in the associated cone. This resolution is called *the short resolution* to distinguish it from the usual minimal free resolution, *the long resolution.*

Let Λ be a generating set of the semigroup which parametrizes the associated algebraic variety. As in [6] we consider a partition of $\Lambda = E \cup A$, where E consists of a chosen generator from each extremal ray. From E we can define the " Apery set" associated with the lattice ideal (see Definition 1.1). The terminology "Apery" comes from the case of numerical semigroups [1].

In section 1, Lemma 1.2 provides a way for computing the Apery set using Gröbner Bases. The first step of the short resolution can be constructed from the Apery set. The second step is described in Proposition 1.4. Now, the complete short resolution can be obtained by the usual methods (for example Schreyer Theorem and its improvements [11]).

In section 2 a combinatorial description of the short resolution is given by means of simplicial complexes. This description is similar to the one which has been used in [7] (see also [12]) for the long resolution.

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In section 3 the main result (Theorem 3.1) of the paper is stated: The regularity of a homogeneous lattice ideal can be obtained from the short resolution. Curiously, in the case of toric curves, the classical techniques of Gruson, Lazarsfeld and Peskine [8] to study the regularity amount to understand the short resolution. The cohomological machinery in [8] is used by L'vovsky in [9] to give an explicit bound for the conductor of a numerical semigroup.

1 Apery sets

Let S be a cancellative commutative semigroup, with zero element and generated by n elements $\Lambda = \{m_1, \dots, m_n\}$. Thus, S is a subsemigroup of a finitely generated abelian group. Denote $G(S)$ the smallest group containing S. The semigroup k-algebra is $k[S] = \bigoplus_{m \in S} k\chi^m$, $(\chi^m \cdot \chi^{m'} = \chi^{m+m'})$. The ideal of S relative to Λ is $\ker(\varphi_0)$, where φ_0 is the k-algebra morphism

$$\varphi_0 : k[\mathbf{X}] \longrightarrow k[S]$$

defined by $\varphi_0(X_i) = \chi^{m_i}$. Notice that φ_0 is surjective, and hence $k[S] \simeq k[\mathbf{X}]/\ker(\varphi_0)$.

Let I be the ideal relative to a fix Λ. Equivalently ([13]), I is a lattice ideal.

Assume that $S \cap (-S) = (0)$. Consider $k[S]$ with the natural S-grading and $k[\mathbf{X}]$ as an S-graded ring, assigning degree m_i to X_i. Notice that I is S-graded because φ_0 is an S-graded morphism of degree zero. The condition $S \cap (-S) = (0)$ says that $k[S]_m$, the homogeneous elements of degree $m \in S$ in $k[S]$, is a k-vector space of finite dimension (see [3]).

Assume that $\mathrm{rank}(G(S)) = d$, let $V = G(S) \bigotimes_{\mathbf{Z}} \mathbf{Q}$, and let $C(S)$ be the cone generated by the image, $\bar{S}$ of S in V. The cone $C(S)$ is strongly convex because $S \cap (-S) = (0)$. Thus, if f is the number of extremal rays of $C(S)$, then $f \geq d$. This implies that there exists a set $E \subset \Lambda$ with $\sharp E = f$, such that $C(E) = C(S)$, where $C(E)$ is the cone in $V(S)$ generated by E. Fix such a set E and $A =: \Lambda \setminus E, \sharp A = n - f = r$.

Definition 1.1. *The Apery set Q of S relative to E is defined as*

$$Q = \{q \in S \ \mid \ q - e \notin S, \ \forall e \in E\}.$$

Denote $k[E]$ the subalgebra of $k[S]$,

$$k[E] = \bigoplus_{m \in S_E} k\chi^m,$$

where S_E is the subsemigroup of S generated by E. Let $k[\mathbf{X}_E]$ the polynomial ring in the f indeterminates associated with E. $k[\mathbf{X}_E]$ can be projected over $k[E]$, it is enough to associate to the indeterminate X_i the symbol χ^{m_i}, for any $m_i \in E$.

$k[S]$ is a $k[E]$-module, and therefore also a $k[\mathbf{X}_E]$-module. The set

$$\{\chi^q \ \mid \ q \in Q\},$$

is a minimal system of generators of $k[S]$ as $k[E]$-module, and therefore, also as $k[\mathbf{X}_E]$-module. Since $k[\mathbf{X}_E]$ is noetherian, Q is a finite set.

Assume, for the sake of simplicity, that

$$E = \{m_1, \ldots, m_f\} \text{ and } A = \{m_{f+1}, \ldots, m_n\}.$$

Fix a total order on the monomials of $k[\mathbf{X}] = k[\mathbf{X}_E, \mathbf{X}_A]$, $X_1 < X_2 < \cdots < X_n$, such that:

1. $\mathbf{X}^\alpha < \mathbf{X}^\beta$, implies $\mathbf{X}^{\alpha+\gamma} < \mathbf{X}^{\beta+\gamma}$, for any α, β and γ;
2. If $f = \sum a_\alpha \mathbf{X}^\alpha \in k[\mathbf{X}]$ has the leading monomial $\mathbf{X}^\beta \notin k[\mathbf{X}_A]$, then $\mathbf{X}^\alpha \notin k[\mathbf{X}_A]$, for any α with $a_\alpha \neq 0$.

For example, we can consider the $lex - inf$ order which is defined

$$\alpha >_{lex-inf} \beta \Longleftrightarrow \alpha <_{lex} \beta,$$

where lex order is the lexicografic order for $X_1 > \cdots > X_n$.

Any order with these properties is not a well-ordering. However, since there exists only a finite number of monomials of S-degree $m \in S$, a Gröbner basis of I can be computed from any S-graded generating set of I. Assume that Γ is the reduced Gröbner basis of I for such a order. Let $\mathcal{B}$ be the set of monomials $\mathbf{X}_A^\alpha$ which are not divisible by any leading monomial of Γ.

Lemma 1.2.

$$Q = \{m \in S \ \mid \ m = \sum_{i=f+1}^{n} \alpha_i m_i, \ \textit{where } \mathbf{X}_A^\alpha \in \mathcal{B}\},$$

and in particular, $\mathcal{B}$ is finite.

Proof. We will use Hironaka division remainder of a monomial by a binomial, a monomial. Moreover, if a monomial $\mathbf{X}^\alpha$ of S-degree $m = \sum_{i=1}^n \alpha_i m_i$, is divided by the reduced Gröbner basis, Γ, the remainder $\mathbf{X}^\beta$ is also of degree m, i.e. $m = \sum_{i=1}^n \beta_i m_i$. Thus, if $\alpha \neq \beta$, we obtain a new writing of m in function of the generators of S.

Let $\mathbf{X}_A^\alpha = X_{f+1}^{\alpha_{f+1}} \cdots X_n^{\alpha_n} \in \mathcal{B}$, and let $m = \sum_{i=f+1}^n \alpha_i m_i$. If m doesn't admit another way of writing in function of the generators, then $m \in Q$ and we are done. Otherwise, $m = \sum_{i=1}^n \beta_i m_i$ for some $\beta_i \in \mathbf{N}$, and $\mathbf{X}_A^\alpha - \mathbf{X}^\beta \in I$. The remainder of $\mathbf{X}_A^\alpha - \mathbf{X}^\beta$ by Γ is zero. Thus, $\mathbf{X}^\beta$ is divisible by some leading monomial of Γ. Therefore, $\mathbf{X}_A^\alpha$ is divisible by some non leading monomial of Γ, which will only have variables corresponding to A. Property 2 of the order guarantees that $\mathbf{X}^\beta \in k[\mathbf{X}_A]$. Thus, $m \in Q$.

Reciprocally, let $m \in Q$. It is possible to write $m = \sum_{i=f+1}^n \alpha_i m_i$, for some $\alpha_i \in \mathbf{N}$. Suppose that $\mathbf{X}_A^\alpha$ is divisible by some leading monomial of Γ. Thus, the remainder of $\mathbf{X}_A^\alpha$ by Γ is $\mathbf{X}^\beta$, which is not divisible by any leading monomial of Γ and it is of degree m. Since $m \in Q$, $\mathbf{X}^\beta = \mathbf{X}_A^\beta \in \mathcal{B}$ and we are done.

$\mathcal{B}$ is finite because Q is finite and any element in S only admits a finite number of writings in function of the generators. ∎

Let l_0 be the cardinality of $\mathcal{B} = \{\mathbf{X}_A^{\alpha_1}, \ldots, \mathbf{X}_A^{\alpha_{l_0}}\}$, and define the $k[\mathbf{X}_E]$-module morphism

$$\Psi_0 : k[\mathbf{X}_E]^{l_0} \longrightarrow k[S],$$

$\Psi_0(e_i) = \mathbf{X}_A^{\alpha_i} + I$, where we are using the isomorphism $k[S] \simeq k[\mathbf{X}]/I$. Equivalentely, $\Psi_0(e_i) = \chi^{q_i}$, where $q_i \in Q$ is the S-degree of the binomial $\mathbf{X}_A^{\alpha_i}$. (It is possible that the cardinality of $\mathcal{B}$ is greater than that of Q, and therefore $q_i = q_j$, for some $i \neq j$.)

Ψ_0 is surjective because $\{\chi^q \mid q \in Q\}$ is a generating set of $k[S]$ as $k[\mathbf{X}_E]$-module.

Any element in Γ whose leading monomial $\mathbf{X}_E^v \mathbf{X}_A^u$ has variables in $\{X_i \mid 1 \leq i \leq f\}$ (i.e. $v \neq 0$), is , except sign $\pm$,

$$\mathbf{X}_E^v \mathbf{X}_A^u - \mathbf{X}_E^{v'} \mathbf{X}_A^{u'}.$$

Property 2 of the order says that $v' \neq 0$, and therefore, since Γ is a reduced Gröbner basis, $\mathbf{X}_A^u$ and $\mathbf{X}_A^{u'} \in \mathcal{B}$. Moreover, $u \neq u'$ because otherwise, since I is a saturated ideal, $\mathbf{X}_E^v - \mathbf{X}_E^{v'} \in I$, which is a contradiction with Γ is a Gröbner basis.

Suppose that $\mathbf{X}_A^u = \mathbf{X}_A^{\alpha_i}$, and $\mathbf{X}_A^{u'} = \mathbf{X}_A^{\alpha_j}$. We associate with the chosen element in Γ, the element in $k[\mathbf{X}_E]^{l_0}$ with all the coordinates equal to zero, except the ith and jth ones, which are $\mathbf{X}_A^u$, and $-\mathbf{X}_A^{u'}$, respectively.

In this way, if l_1 is the number of element in Γ of the above form, we obtain $G_i \in k[\mathbf{X}_E]^{l_0}$, $1 \leq i \leq l_1$. Let $\mathcal{N}$ be the matrix

$$\mathcal{N} = (G_1 | \ldots | G_{l_1}).$$

$\mathcal{N}$ defines a morphism of free $k[\mathbf{X}_E]$-modules

$$\Psi_1 : k[\mathbf{X}_E]^{l_1} \longrightarrow k[\mathbf{X}_E]^{l_0}.$$

Proposition 1.3.

$$\mathit{coker}(\mathcal{N}) \simeq_{k[\mathbf{X}_E]} k[S].$$

Proof. Since $\mathit{coker}(\mathcal{N}) \simeq k[\mathbf{X}_E]^{l_0}/\mathit{im}\Psi_1$ and $k[S] \simeq k[\mathbf{X}_E]^{l_0}/\mathit{ker}\Psi_0$, it is enough to prove that

$$\mathit{im}\Psi_1 = \mathit{ker}\Psi_0.$$

It is clear that $\mathit{im}\Psi_1 \subset \mathit{ker}\Psi_0$. Let $(F_1(\mathbf{X}_E), \ldots, F_{l_0}(\mathbf{X}_E)) \in \mathit{ker}\Psi_0$ be a homogeneous element of degree $m \in S$. This means that if $F_i \neq 0$, then F_i is homogeneous of degree $m - q_i$, where q_i is the S-degree of $\mathbf{X}_A^{\alpha_i}$. The element $F = \sum_{i=1}^{l_0} F_i \mathbf{X}_A^{\alpha_i} \in I$, and therefore the remainder of the Hironaka division of F by Γ is 0.

Notice that if $F_i \neq 0$ then $F_i \notin k$; otherwise, $\mathbf{X}_A^{\alpha_i}$ would appear in the remainder. Thus, $F = 0$ and we are done, or the leading monomial of F is $\pm \mathbf{X}_E^{\omega} \mathbf{X}_A^{\alpha_i}$ for some i, and $\omega \neq 0$. In the latter case, there exists

$$\pm(\mathbf{X}_E^v \mathbf{X}_A^{\alpha_i} - \mathbf{X}_E^{v'} \mathbf{X}_A^{\alpha_j}) \in \Gamma,$$

where $\omega = \upsilon + \beta$, and $\upsilon \neq 0$. Assume, for simplicity's sake, that both elements are positive.

Consider

$$F^{(1)} := F - \mathbf{X}_E^{\beta}(\mathbf{X}_E^{\upsilon}\mathbf{X}_A^{\alpha_i} - \mathbf{X}_E^{\upsilon'}\mathbf{X}_A^{\alpha_j}).$$

Notice that the leading monomial of $F^{(1)}$ is less than the leading monomial of F. We can write

$$F^{(1)} = \sum_{l=1}^{l_0} F_l^{(1)} \mathbf{X}_A^{\alpha_l} \in I,$$

where the following equations are satisfied:

$$\begin{aligned} F_i^{(1)} &= F_i - \mathbf{X}_E^{\omega}, \\ F_j^{(1)} &= F_j + \mathbf{X}_E^{\beta+\upsilon'}, \\ F_l^{(1)} &= F_l, \quad \text{for all} \quad l \neq i, j. \end{aligned}$$

Equivalently, if we suppose $i < j$ and denote

$$G^{(1)} := (0, \ldots, 0, \mathbf{X}_E^{\upsilon}, 0, \ldots, 0, -\mathbf{X}_E^{\upsilon'}, 0, \ldots, 0) \in \{G_1, \ldots, G_{l_1}\},$$

we obtain

$$(F_1^{(1)}, \ldots, F_{l_0}^{(1)}) = (F_1, \ldots, F_{l_0}) - \mathbf{X}_E^{\beta} G^{(1)}.$$

If $F^{(1)} = 0$, then $F_l^{(1)} = 0$ for any l, and we are done.

If $F^{(1)} \neq 0$, we can proceed by recurrence. Suppose that for, fix $r \geq 2$ and for any j, $1 \leq j \leq r-1$, there exists

$$F^{(j)} = F_1^{(j)}(\mathbf{X}_E)\mathbf{X}_A^{\alpha_1} + \cdots + F_{l_0}^{(j)}(\mathbf{X}_E)\mathbf{X}_A^{\alpha_{l_0}} \in I - \{0\},$$

satisfying

$$(F_1^{(j)}, \ldots, F_{l_0}^{(j)}) = (F_1^{(j-1)}, \ldots, F_{l_0}^{(j-1)}) - \mathbf{X}_E^{\beta^{(j)}} G^{(j)},$$

where $G^{(j)} \in \{G_1, \ldots, G_{l_1}\}$, and $\beta^{(j)} \in \mathbf{N}^n$ $(\beta^{(1)} = \beta)$.

Reasoning as before, we obtain $G^{(r)} \in \{G_1, \ldots, G_{l_1}\}$ and $\mathbf{X}_E^{\beta^{(r)}}$ such that if

$$(F_1^{(r)}, \ldots, F_{l_0}^{(r)}) = (F_1^{(r-1)}, \ldots, F_{l_0}^{(r-1)}) - \mathbf{X}_E^{\beta^{(r)}} G^{(r)},$$

then

$$F^{(r)} = \sum_{l=1}^{l_0} F_l^{(r)} \mathbf{X}_A^{\alpha_l} \in I,$$

and the leading monomial of $F^{(r)}$ is less than the leading monomial of $F^{(r-1)}$.

Therefore, if $F^{(r)} = 0$, we obtain

$$(F_1^{(r-1)}, \ldots, F_{l_0}^{(r-1)}) = \mathbf{X}_E^{\beta^{(r)}} G^{(r)}.$$

From

$$(F_1^{(r-1)},\dots,F_{l_0}^{(r-1)}) = (F_1,\dots,F_{l_0}) - \sum_{i=1}^{r-2} \mathbf{X}_E^{\beta^{(i)}} G^{(i)},$$

we obtain

$$(F_1,\dots,F_{l_0}) = \sum_{i=1}^{r-1} \mathbf{X}_E^{\beta^{(i)}} G^{(i)},$$

and we are done.

If $F^{(r)} \neq 0$ the result follows by recurrence, because the elements $F^{(j)}$ are homogeneous of degree m, and in each step the leading monomial decreases.

□

Therefore, we obtain the first step of a free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module that is S-graded:

$$k[\mathbf{X}_E]^{l_1} \xrightarrow{\Psi_1} k[\mathbf{X}_E]^{l_0} \xrightarrow{\Psi_0} k[S] \rightarrow 0.$$

In the above proof, for any element $\mathbf{X}_A^{\alpha_i} \in \mathcal{B}$ we have considered its S-degree $q_i \in Q$. Assume, for simplicity's sake, that $Q = \{q_1,\dots,q_{\beta_0}\}$, where $\beta_0 = \sharp Q$. Notice that $\beta_0 \leq l_0$. In the case $\beta_0 < l_0$, if $\beta_0 + 1 \leq i \leq l_0$, $q_i = q_j$ for a unique j, $1 \leq j \leq \beta_0$. Denote $j = j(i)$. We consider

$$\pi : k[\mathbf{X}_E]^{l_0} \rightarrow k[\mathbf{X}_E]^{\beta_0},$$

the $k[\mathbf{X}_E]$-module morphism defined by

$$\pi(e_i) = \begin{cases} e_i & \text{if } 1 \leq i \leq \beta_0 \\ e_{j(i)} & \text{if } \beta_0 + 1 \leq i \leq l_0 \end{cases}$$

Notice that $\pi \circ \Psi_1 : k[\mathbf{X}_E]^{l_1} \rightarrow k[\mathbf{X}_E]^{\beta_0}$, is given by the matrix

$$\mathcal{M} := (\pi(G_1)|\dots|\pi(G_{l_1})).$$

On the other hand, considering the morphism of $k[\mathbf{X}_E]$-modules

$$\Phi_0 : k[\mathbf{X}_E]^{\beta_0} \longrightarrow k[S],$$

defined by $\Phi_0(e_i) = \chi^{q_i}$, $1 \leq i \leq \beta_0$. It is clear that $\Phi_0 \circ \pi = \Psi_0$.

Proposition 1.4.

$$coker(\mathcal{M}) \simeq_{k[\mathbf{X}_E]} k[S].$$

Proof. The situation is the following

$$\begin{array}{ccccc} k[\mathbf{X}_E]^{l_1} & \xrightarrow{\Psi_1} & k[\mathbf{X}_E]^{l_0} & \xrightarrow{\Psi_0} & k[S] \\ & {\scriptstyle \pi\circ\Psi_1}\searrow & {\scriptstyle \pi}\downarrow & {\scriptstyle \Phi_0}\nearrow & \\ & & k[\mathbf{X}_E]^{\beta_0} & & \end{array}$$

As in Proposition 1.3, it is enough to prove that

$$im(\pi \circ \Psi_1) = ker(\Phi_0).$$

$im(\pi \circ \Psi_1) \subset ker(\Phi_0)$ follows from $\Phi_0(\pi(G_i)) = \Psi_0(G_i) = 0$, for any i, $1 \leq i \leq l_1$.

For the other inclusion, let $(F_1(\mathbf{X}_E), \ldots, F_{\beta_0}(\mathbf{X}_E)) \in ker\Phi_0$. Thus,

$$(F_1(\mathbf{X}_E), \ldots, F_{\beta_0}(\mathbf{X}_E), 0, \ldots, 0) \in ker\Psi_0 = im\Psi_1.$$

There exist $\lambda_i(\mathbf{X}_E)$ such that $(F_1, \ldots, F_{\beta_0}, 0, \ldots, 0) = \sum_{i=1}^{l_1} \lambda_i G_i$. Notice that, if we denote G_{ij} the jth coordinate of G_i, then

$$\sum_{i=1}^{l_1} \lambda_i G_{ij} = 0,$$

for any j, $\beta_0 + 1 \leq j \leq l_0$. Therefore, if we denote $\pi(G_i)_t$ the t-th coordinate of $\pi(G_i)$, notice that

$$\pi(G_i)_t = G_{it} + \sum_{j(s)=t,\ \beta_0+1 \leq s \leq l_0} G_{is}, \quad 1 \leq t \leq \beta_0.$$

Thus, for a fix t,

$$\sum_{i=1}^{l_1} \lambda_i \pi(G_i)_t = \sum_{i=1}^{l_1} \lambda_i G_{it}.$$

Therefore,

$$(F_1, \ldots, F_{\beta_0}) = \sum_{i=1}^{l_1} \lambda_i \pi(G_i),$$

and we are done. □

From the free resolution

$$k[\mathbf{X}_E]^{l_1} \stackrel{\pi \circ \Psi_1}{\to} k[\mathbf{X}_E]^{\beta_0} \stackrel{\Phi_0}{\to} k[S] \to 0,$$

using the Schreyer Theorem and its improvements (see [11]), the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module can be obtained. We will call this resolution, *the short resolution* of $k[S]$ to distinguish it from the minimal free resolution of $k[S]$ as $k[\mathbf{X}]$-module.

2 Combinatorial description of the short resolution

Assume that $S \neq (0)$, and consider the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module

$$0 \to k[\mathbf{X}_E]^{\beta_{f-1}} \stackrel{\Phi_{f-1}}{\to} \cdots \to k[\mathbf{X}_E]^{\beta_2} \stackrel{\Phi_2}{\to} k[\mathbf{X}_E]^{\beta_1} \stackrel{\Phi_1}{\to} k[\mathbf{X}_E]^{\beta_0} \stackrel{\Phi_0}{\to} k[S] \to 0.$$

The S-graded Nakayama's lemma (see [3]) says this resolution is unique except isomorphisms. Moreover, denoting $M_i = \ker(\Phi_i)$ the ith module of syzygies of $k[S]$ as $k[\mathbf{X}_E]$-module, $0 \leq i \leq f-1$, we obtain

$$\beta_{i+1} = \sum_{m\in S} \dim W_i(m),$$

where $W_i(m) := (M_i/\mathfrak{m}_E M_i)_m$ is considered as a k-vector space, and $\mathfrak{m}_E$ is the ideal of $k[\mathbf{X}_E]$ generated by the indeterminates of $\mathbf{X}_E$ (X_i such that $m_i \in E$).

We will show how this resolution can be described by means of some simplicial complexes. Concretely, if $m \in S$, let T_m be the simplicial complex

$$T_m = \{F \subset E \quad | \quad m - n_F \in S\}.$$

Denote $\widetilde{H}_i(T_m)$ the ith reduced homology space of the simplicial complex T_m, and let $\tilde{h}_i(T_m) = \dim(\tilde{H}_i(T_m))$.

Proposition 2.1.

$$\tilde{H}_i(T_m) \simeq W_i(m),$$

for any $m \in S$ and for any i, $0 \leq i \leq f-2$.

Proof. Let us consider $k[S]$ and $k \simeq k[\mathbf{X}_E]/\mathfrak{m}_E$ as $k[\mathbf{X}_E]$-modules and use the commutativity of the functor Tor, concretely

$$\mathrm{Tor}_{i+1}(k[S], k) \simeq \mathrm{Tor}_{i+1}(k, k[S]).$$

In order to compute the space $\mathrm{Tor}_{i+1}(k[S], k)$ as $k[\mathbf{X}_E]$-module, take the Koszul complex for the regular sequence $\{X_i \quad | \quad m_i \in E\}$, which is an exact sequence.

$$0 \to \bigwedge^{f} k[\mathbf{X}_E]^f \overset{d_{f-1}}{\to} \cdots \to \bigwedge^{j+1} k[\mathbf{X}_E]^f \overset{d_j}{\to} \bigwedge^{j} k[\mathbf{X}_E]^f \overset{d_{j-1}}{\to} \cdots \to k[\mathbf{X}_E]^f \overset{d_0}{\to} k[\mathbf{X}_E] \to k \to 0.$$

Here d_j is given by

$$d_j(e_{i_0} \wedge \cdots \wedge e_{i_j}) = \sum_{l=0}^{j} (-1)^l X_l \; e_{i_0} \wedge \cdots \wedge e_{i_{l-1}} \wedge e_{i_{l+1}} \wedge \cdots \wedge e_{i_j}.$$

These homomorphism are S-graded of degree 0 assigning the degree $m_{i_0} + \cdots + m_{i_j}$ to the element $e_{i_0} \wedge \cdots \wedge e_{i_j}$. Tensoring this exact sequence with the $k[\mathbf{X}_E]$-module $k[S]$, we obtain the S-graded Koszul complex

$$0 \to \bigwedge^{f} k[S]^f \to \cdots \to \bigwedge^{j+1} k[S]^f \overset{d_j}{\to} \bigwedge^{j} k[S]^f \overset{d_{j-1}}{\to} \cdots \to k[S]^f \overset{d_0}{\to} k[S] \to k \to 0.$$

The restriction to its degree $m \in S$ is the following complex of finite-dimensional k-vector space

$$\cdots \to \bigoplus_{\substack{F\subset E \\ \sharp F = 3}} k[S]_{m-n_F} \to \bigoplus_{\substack{F\subset E \\ \sharp F = 2}} k[S]_{m-n_F} \to \bigoplus_{\substack{F\subset E \\ \sharp F = 1}} k[S]_{m-n_F} \to k[S]_m \to 0.$$

Notice that this complex can be identified with the augmented oriented chain complex of T_m, because

$$k[S]_{m-n_F} = \begin{cases} k, & \text{if } F \in T_m \\ 0, & \text{otherwise} \end{cases}$$

Thus, we obtain that

$$(\mathrm{Tor}_{i+1}(k[S], k))_m \simeq \widetilde{H}_i(T_m).$$

In order to compute $\mathrm{Tor}_{i+1}(k, k[S])$ as $k[\mathbf{X}_E]$-modules, take the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module. Tensoring with $k \simeq k[\mathbf{X}_E]/\mathfrak{m}_E$ it is obtained

$$0 \to (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_{f-1}} \stackrel{\tilde{\Phi}_{f-1}}{\to} \cdots \to (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_2} \stackrel{\tilde{\Phi}_2}{\to} (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_1} \stackrel{\tilde{\Phi}_1}{\to} (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_0} \to 0.$$

Thus, $(\mathrm{Tor}_{i+1}(k, k[S]))_m \simeq W_i(m)$.

Now it is clear that the isomorphism follows from the commutativity of the functor Tor.

□

As an application of these isomorphisms, if denote

$$D(i) := \{m \in S \;\mid\; \widetilde{H}_i(T_m) \neq 0\},$$

we obtain that

$$\beta_{i+1} = \sum_{i \in D(i)} \widetilde{h}_i(T_m),\ -1 \leq i \leq f-2.$$

Notice that, by the noetherian property, $D(i)$ is finite. Moreover, we can state the following corollary.

Corollary 2.2. *In the above setting, the sets $D(i)$ can be computed from the short resolution.*

Remark 2.3. *The results in section 1 allow the computation of the sets $D(i)$ using Gröbner Bases. This method is more useful for explicit computations than that in [4] using Hilbert bases of some diophantine systems (see [10]).*

3 The regularity of a homogeneous lattice ideal

Assume that I is a homogeneous ideal for the natural grading. In this case, it is well defined $||m|| = ||\alpha||_1$, where $m = \sum_{i=1}^n \alpha_i m_i$ and $||\alpha||_1 = \sum_{i=1}^n \alpha_i$.

Theorem 3.1. *The regularity of a homogeneous lattice ideal I can be computed from the short resolution.*

Proof. It is enough to use 2.2 and the following formula ([4])

$$reg(I) = max_{-1 \leq i \leq f-2}\{u_i - i\},$$

where $u_i = max\{||m|| \mid m \in D(i)\}$. □

Example 3.2. Consider the semigroup $S \subset \mathbb{N}^3$ generated by

$$[5,0,0],[0,5,0],[0,0,5],[[4,1,0],[1,4,0],[2,3,0],[0,1,4],[0,4,1],[0,2,3].$$

A projective simplicial toric surface.

The Gröbner basis of I respect lex-inf is

$$\begin{aligned}\Gamma = \{ \quad & -x_3x_8+x_9^2, x_9x_8-x_2x_7, x_8^3-x_2^2x_9, -x_2x_3+x_7x_8, \\ & -x_9x_3+x_7^2, -x_2x_4+x_6^2, x_5^2-x_6x_2, -x_1x_5+x_6x_4, \\ & x_4x_5-x_1x_2, -x_1^2x_6+x_4^3, \underline{-x_2x_7x_9+x_3x_8^2}, \underline{x_2x_4^2-x_6x_1x_5}\}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{B} = \{ \quad & 1, x_4, x_4^2, x_5, x_6, x_5x_6, x_7, x_4x_7, x_4^2x_7, \\ & x_5x_7, x_6x_7, x_5x_6x_7, x_8, x_4x_8, x_4^2x_8, \\ & x_5x_8, x_6x_8, x_5x_6x_8, x_8^2, x_4x_8^2, x_4^2x_8^2, x_5x_8^2, \\ & x_6x_8^2, x_5x_6x_8^2, x_9, x_4x_9, x_4^2x_9, x_5x_9, x_6x_9, x_5x_6x_9, \\ & x_7x_9, x_4x_7x_9, x_4^2x_7x_9, x_5x_7x_9, x_6x_7x_9, x_5x_6x_7x_9\},\end{aligned}$$

and

$$\begin{aligned}Q = \{ \quad & [0,0,0],[4,1,0],[8,2,0],[1,4,0],[2,3,0],[3,7,0],[0,1,4],[4,2,4],[8,3,4], \\ & [1,5,4],[2,4,4],[3,8,4],[0,4,1],[4,5,1],[8,6,1],[1,8,1],[2,7,1],[3,11,1], \\ & [0,8,2],[4,9,2],[8,10,2],[1,12,2],[2,11,2],[3,15,2],[0,2,3],[4,3,3],[8,4,3], \\ & [1,6,3],[2,5,3],[3,9,3],[0,3,7],[4,4,7],[8,5,7],[1,7,7],[2,6,7],[3,10,7]\},\end{aligned}$$

and therefore $u_{-1} = 4$. Notice that $\sharp\mathcal{B} = \sharp Q = 36$.

From the two underlined binomial in Γ we obtain $D(0) = \{(8,7,0),(0,8,7)\}$, and therefore $u_0 = 3$.

The short resolution is

$$0 \to k[\mathbf{X}_E]^2 \stackrel{\Phi_1}{\to} k[\mathbf{X}_E]^{36} \stackrel{\Phi_0}{\to} k[S] \to 0.$$

The regularity of I is

$$reg(I) = max\{u_0 - 0 = 3, u_{-1} + 1 = 5\} = 5.$$

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