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The hull Resolution of a monomial curve in $\mathbb{A}^3(k)$.

Ignacio Ojeda Martínez de Castilla, Pilar Pisón Casares

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Departamento de Álgebra. Universidad de Sevilla

The hull Resolution of a monomial curve in $\mathbb{A}^3(k)$ *

Ignacio Ojeda Martínez de Castilla
Dpto. de Matemáticas. Universidad de Extremadura
ojedamc@unex.es
Pilar Pisón Casares
Dpto. de Álgebra. Universidad de Sevilla
ppison@cica.es

Abstract

We characterize the hull resolution of a monomial curve in the three dimensional affine space and compare it with its minimal free resolution. Concretely, we give a necessary and sufficient condition for which the hull resolution is minimal in terms of the semigroup associated with.

Introduction

Let $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ be the polynomial ring in n variables over a field k. Throughout this paper $\mathbf{x}^{\mathbf{u}}$ will denote the monomial $x_1^{u_1} \cdots x_n^{u_n}$ with $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{Z}_0^n$.

The hull resolution of the $\mathbb{Z}^n/\mathcal{L}$-graded lattice ideal

$$I_{\mathcal{L}} := \langle\ \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u} - \mathbf{v} \in \mathcal{L} \text{ with } \mathbf{u}, \mathbf{v} \in \mathbb{Z}_0^n \rangle,$$

where $\mathcal{L} \subseteq \mathbb{Z}^n$ is a $\mathbb{Z}$-module such that $\mathcal{L} \cap \mathbb{Z}_0^n = \{\mathbf{0}\}$, was introduced by D. Bayer and B. Sturmfels in [2]. In that work, the authors construct a new canonical free resolution of $I_{\mathcal{L}}$ from an unbounded convex polyhedron P_t (originally introduced by I. Barany, R. Howe and H. Scarf in [1]) and a regular cell complex X (cf. [4] pp. 253–255).

The hull resolution of a lattice ideal is far from being minimal, but, unlike minimal resolutions, it respects symmetry and preserves the action on $I_{\mathcal{L}}$ by the lattice $\mathcal{L}$. Furthermore, the involved free modules are of finite rank over $k[\mathbf{x}]$ and there are finitely many of them. This makes interesting the comparison of minimal and hull resolutions of lattice ideals in order to decide when they agree.

In this paper, we center our attention in a particular class of lattice ideals. We only consider the ideals defining monomial curves in the 3-dimensional affine space. From a new and explicit description of the minimal resolution in terms of combinatorial arguments (Theorem 2.3), we obtain a complete characterization of the hull resolution of a monomial curve in $\mathbb{A}^3(k)$:

Main Theorem. *Let $I \subset k[\mathbf{x}]$ be an ideal defining a monomial curve in the 3-dimensional affine space. The hull resolution of I is*

$$0 \longrightarrow k[\mathbf{x}] \longrightarrow k[\mathbf{x}]^2 \longrightarrow k[\mathbf{x}] \longrightarrow k[\mathbf{x}]/I \longrightarrow 0,$$

if $\langle x_i^{\alpha_i} - x_j^{\alpha_j}, x_k^{\alpha_k} - (x_i x_j)^{\gamma} \rangle$ is a minimal system of generator of I, for some threesome $\{i, j, k\} = \{1, 2, 3\}$. Otherwise the hull resolution of I is

$$0 \longrightarrow k[\mathbf{x}]^2 \longrightarrow k[\mathbf{x}]^3 \longrightarrow k[\mathbf{x}] \longrightarrow k[\mathbf{x}]/I \longrightarrow 0.$$

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As a corollary we give a necessary and sufficient condition for which the hull resolution of a monomial curve in $\mathbb{A}^3(k)$ is minimal in terms of the semigroup associated with.

Finally, we would like to emphasize that the study of the connections between semigroups and lattice ideals is an active research field as it can be seen through the abundant literature about it (for more details see [5]).

1 The hull resolution of a lattice ideal

Let $\mathcal{L} \subseteq \mathbb{Z}^n$ be a *lattice*, that is a finitely generated subgroup of $\mathbb{Z}^n$, such that $\mathcal{L} \cap \mathbb{Z}^n_0 = \{\mathbf{0}\}$.

We write $M_{\mathcal{L}}$ for the *lattice module* generated by $\{\mathbf{x}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\}$, in other words, the monomial $k[\mathbf{x}]$-submodule of the Laurent polynomial ring, $k[\mathbf{x}^{\pm}] := k[\mathbf{x}][x_1^{-1}, \ldots, x_n^{-1}]$

$$M_{\mathcal{L}} := k[\mathbf{x}]\{\mathbf{x}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\} = k\{\mathbf{x}^{\mathbf{u}} \mid \mathbf{u} \in \mathbb{Z}^n_0 + \mathcal{L}\} \subset k[\mathbf{x}^{\pm}].$$

The hypothesis $\mathcal{L} \cap \mathbb{Z}^n_0 = \{\mathbf{0}\}$ assures that the elements in $M_{\mathcal{L}}$ with exponent in $\mathcal{L}$ form a minimal system of generators of $M_{\mathcal{L}}$ in the sense of [2]. Moreover, the lattice $\mathcal{L}$ acts on the lattice module $M_{\mathcal{L}}$; the $\mathcal{L}$-action is given by $\mathbf{x}^{\mathbf{v}} + \mathbf{b} = \mathbf{x}^{\mathbf{v}+\mathbf{b}}$ with $\mathbf{b} \in \mathcal{L}$ and $\mathbf{x}^{\mathbf{v}} \in M_{\mathcal{L}}$.

For $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ and $t \in \mathbb{R}_+$ we abbreviate $\mathrm{t}^{\mathbf{v}} = (t^{v_1}, \ldots, t^{v_n}) \in \mathbb{R}^n_+$. Fix any real number t larger that $(n+1)! = 2 \cdot 3 \cdots (n+1)$. We define P_t to be the convex hull of the point set $\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{x}^{\mathbf{v}} \in M_{\mathcal{L}}\} \subset \mathbb{R}^n_+$.

Remark 1.1. The set P_t is an unbounded n-dimensional convex polyhedron.

From Lemma 2.1 in [2] it follows that $P_t = \mathbb{R}^n_+ + \mathrm{conv}\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\} \subseteq \mathbb{R}^n_+$, and, by Proposition 2.2 in [2], one has that the vertices of P_t are precisely the points $\mathrm{t}^{\mathbf{v}}$ with $\mathbf{v} \in \mathcal{L}$.

Lemma 1.2. *(Theorem 2.3 in [2]) The face poset of the polyhedron P_t is independent of t for $t > (n+1)!$. The same holds for the subposet of all bounded faces of P_t.*

Definition 1.3. *The* hull complex *of $M_{\mathcal{L}}$, denoted* $\mathrm{Hull}(M_{\mathcal{L}})$, *is the regular cell complex, equipped with a choice of an* incidence function ε, *of bounded faces of P_t for large t.*

For simplicity, in the following we will write X for the hull complex $\mathrm{Hull}(M_{\mathcal{L}})$.

The hull complex X inherits a $\mathbb{Z}^n$-grading from the generators of $M_{\mathcal{L}}$ as follows. Let F be a nonempty face of X. We identify F with its set of vertices $\{\mathrm{t}^{\mathbf{v}_1}, \ldots, \mathrm{t}^{\mathbf{v}_r}\}$, a finite subset of $\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\}$. Set $m_F := \mathrm{lcm}(\mathbf{x}^{\mathbf{v}_1}, \ldots, \mathbf{x}^{\mathbf{v}_r})$. The exponent vector of the monomial m_F is the *join* $\mathbf{v}_F := \vee\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ in $\mathbb{Z}^n$. We call $\mathbf{v}_F$ the *degree* of the face F.

On the other hand, the $\mathcal{L}$-action on $M_{\mathcal{L}}$ and a suitable choice of the incidence function ε (cf. proof of Theorem 3.9 in [2]) assure that the hull complex X is *equivariant*, that is, $F \in X$ and $\mathbf{b} \in \mathcal{L}$, implies $F + \mathbf{b} \in X$ (X is $\mathcal{L}$-invariant) and the incidence function ε satisfies $\varepsilon(F, F') = \varepsilon(F + \mathbf{b}, F' + \mathbf{b})$, for every $\mathbf{b} \in \mathcal{L}$.

Definition 1.4. *The chain complex $\mathbf{F}_X$ is the $\mathbb{Z}^n$-graded $k[\mathbf{x}]$-module*

$$\mathbf{F}_X = \bigoplus_{F \in X,\ F \neq \emptyset} k[\mathbf{x}] \cdot e_F, \quad \textit{with differential } \partial\, e_F := \sum_{F' \in X,\ F' \neq \emptyset} \varepsilon(F, F') \frac{m_F}{m_{F'}}\, e_{F'}.$$

Theorem 1.5. *(Theorem 2.5 in [2]) The chain complex $\mathbf{F}_X$ is a free resolution of $M_{\mathcal{L}}$, called the* hull resolution *of $M_{\mathcal{L}}$.*

Now we are at the disposal to define the hull resolution of a lattice ideal $I_{\mathcal{L}}$. Following the results in [2] section 3, one has that the hull resolution of $I_{\mathcal{L}}$ is the image by a certain functor of the hull resolution of $M_{\mathcal{L}}$. In this section we will define the hull resolution of $I_{\mathcal{L}}$ as in [2] but avoiding any reference to the functorial equivalence.

The group $\mathcal{L}$ acts on the faces of X. Let $X/\mathcal{L}$ denote the set of orbits. For each orbit $\mathcal{F} \in X/\mathcal{L}$ we select a distinguished representative $\mathrm{Rep}(F) \in \mathcal{F}$ such that $\mathrm{t}^{\mathbf{0}}$ is adjoined to F, and we write $\mathrm{Rep}(X/\mathcal{L})$ for the set of representatives, by Lemma 3.13 in [2] one has that this set is finite.

Definition 1.6. *The chain complex* $\mathbf{F}_X^*$ *is the* $\mathbb{Z}^n/\mathcal{L}$*-graded* $k[\mathbf{x}]$*-modulo*

$$\mathbf{F}_X^* = \bigoplus_{F\in \mathrm{Rep}(X/\mathcal{L}),\ F\neq\emptyset} k[\mathbf{x}]\cdot \mathbf{v}_F$$

with differential $\partial^* v_F := \mathrm{Rep}(\partial\ \mathbf{v}_F)$*, where* ∂ *is the differential of* $\mathbf{F}_X$ *and*

$$\mathrm{Rep}: \bigoplus_{F\in X,\ F\neq\emptyset} k[\mathbf{x}]\cdot \mathbf{v}_F \longrightarrow \bigoplus_{F\in \mathrm{Rep}(X/\mathcal{L}),\ F\neq\emptyset} k[\mathbf{x}]\cdot \mathbf{v}_F$$

is the $k[\mathbf{x}]$*-modulo map given by* $\mathrm{Rep}(\mathbf{v}_F) = \mathbf{v}_{\mathrm{Rep}(F)}$.

Theorem 1.7. *([2]) The chain complex* $\mathbf{F}_X^*$ *is a free resolution of* $I_{\mathcal{L}}$*, called the* hull resolution *of* $I_{\mathcal{L}}$.

2 The minimal resolution of a monomial curve in $\mathbb{A}^3(k)$.

Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbb{Z}_+,\ i = 1,2,3$, and $\gcd(n_1,n_2,n_3) = 1$, and let $G(S) \subset \mathbb{Z}$ be the group generated by S.

We consider $\mathbf{u}_1 = (1,0,0),\ \mathbf{u}_2 = (0,1,0)$ and $\mathbf{u}_3 = (0,0,1)$ in $\mathbb{Z}^3$, and the $\mathbb{Z}$-linear surjective map $\pi : \mathbb{Z}^3 \longrightarrow G(S)$, where $\pi(\mathbf{u}_i) = n_i,\ i = 1,2,3$. We write $\mathcal{L}$ for the kernel of π,

$$\mathcal{L} := \ker \pi = \{\mathbf{v} = (v_1,v_2,v_3) \in \mathbb{Z}^3 \mid \sum_{i=1}^{3} v_i n_i = 0\}$$

Obviously $\mathcal{L} \subseteq \mathbb{Z}^3$ is a lattice such that $\mathcal{L}\cap\mathbb{Z}_0^3 = \{\mathbf{0}\}$. Thus, we have that the ideal of the affine monomial curve $\{(\lambda^{n_1},\lambda^{n_2},\lambda^{n_3}) \mid \lambda\in k\}$ is the lattice ideal $I_{\mathcal{L}}$ (cf. [6]).

Remark 2.1. Since the lattice $\mathcal{L}$ is defined from the semigroup S, in the following we will write I_S for $I_{\mathcal{L}}$.

We define $\alpha_1 \in \mathbb{Z}_+$ to be the least positive integer such that $\alpha_1 n_1 \in \mathbb{Z}_0 n_2 + \mathbb{Z}_0 n_3$ and α_2 and α_3 analogously. That choice of α_1, α_2 and α_3 implies the existence of γ_{ij} and $\gamma_{ik} \in \mathbb{Z}_0$ (not uniquely defined) such that $\alpha_i n_i = \gamma_{ij} n_j + \gamma_{ik} n_k$, for each threesome $\{i,j,k\} = \{1,2,3\}$.

Theorem 2.2. *([6, 3]) With the notation introduced above:*

(a) I_S *is complete intersection (equivalently* S *is symmetric) if and only if there exist* $i,j\in\{1,2,3\},\ i\neq j$ *such that* $\alpha_i n_i = \alpha_j n_j$*. In this case, the only minimal binomial systems of generators (except unity in* $k[\mathbf{x}]$*) is*

$$F_1 = x_i^{\alpha_i} - x_j^{\alpha_j},\ F_2 = x_k^{\alpha_k} - x_i^{\gamma_{ki}} x_j^{\gamma_{kj}},$$

for some threesome $\{i,j,k\} = \{1,2,3\}$*. Moreover, if* $\alpha_k n_k \neq \alpha_i n_i$*, then such a threesome is unique.*

(b) I_S *is not complete intersection (equivalently* S *is not symmetric) if and only if* γ_{ki}, γ_{kj} *are both not zero for every threesome* $\{i,j,k\} = \{1,2,3\}$*. In this case, one has that the pairs* $\{\gamma_{ki},\gamma_{kj}\}$ *are unique. Moreover, the only minimal binomial system of generators (except unity in* $k[\mathbf{x}]$*) is*

$$F_1 = x_1^{\alpha_1} - x_2^{\gamma_{12}} x_3^{\gamma_{13}},\ F_2 = x_2^{\alpha_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}},\ F_3 = x_3^{\alpha_3} - x_1^{\gamma_{31}} x_2^{\gamma_{32}},$$

where $0 < \gamma_{ki} < \alpha_i,\ i = 1,2,3$ *and* $k \neq i$.

The explicit description of the minimal generating sets of I_S in above theorem can be found in [6], and the uniqueness can be deduced from the combinatorial description of these sets (cf. [3]) by means of some simplicial complexes associated with the elements in the semigroup. Concretely, if $m\in S$ the set

$$\Delta_m := \{F \subseteq \{1,2,3\} \mid m - \sum_{i\in F} n_i \in S\}$$

is a simplicial (abstract) complex. The ith-reduced homology of this complex with values in k is denoted by $\widetilde{H}_i(\Delta_m)$, and $\widetilde{h}_i(\Delta_m)$ is its dimension as a k-vector space.

Let $k[S] \simeq k[\mathbf{x}]/I_S$ be the k-algebra associated with the semigroup, and

$$\Phi_0 : k[\mathbf{x}] \longrightarrow k[S],$$

the presentation map.

Theorem 2.3. *With the same notation as above*

(a) If I_S is complete intersection (equivalently S is symmetric) the minimal free resolution of I_S is:

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{\Phi_2} k[\mathbf{x}]^2 \xrightarrow{\Phi_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Moreover, Φ_1 and Φ_2 can be represented respectively by the matrices

$$A_1 = (F_1\ F_2) \quad \textit{and} \quad A_2 = \begin{pmatrix} F_1 \\ -F_2 \end{pmatrix},$$

where the F_1 and F_2 denote the binomials defined in Theorem 2.2(a).

(b) If I_S is not complete intersection (equivalently S is not symmetric) the minimal free resolution of I_S is:

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{\Phi_2} k[\mathbf{x}]^3 \xrightarrow{\Phi_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Moreover, Φ_1 and Φ_2 can be represented respectively by the matrices

$$A_1 = (F_1\ F_2\ F_3) \quad \textit{and} \quad A_2 = \begin{pmatrix} x_2^{\gamma_{32}} & x_3^{\gamma_{23}} \\ x_3^{\gamma_{13}} & x_1^{\gamma_{31}} \\ x_1^{\gamma_{21}} & x_2^{\gamma_{12}} \end{pmatrix}$$

where the F_1, F_2 and F_3 denote the binomials defined in Theorem 2.2(b).

Proof. First statement follows from the particular form of the binomials F_1 and F_2.

In order to prove the second one, it suffices to see that the first syzygy module of I_S, $N_1 := \ker \Phi_1$, is generated by the column vectors of A_2. For this, we will use that the S-degree, m, of a minimal generating syzygy of I_S satisfies $\widetilde{h}_1(\Delta_m) \neq 0$ (cf. [3]). In this case Δ_m is an empty triangle, equivalently

$$\begin{cases} m - (n_1 + n_2) \in S; \\ m - (n_1 + n_3) \in S; \\ m - (n_2 + n_3) \in S; \\ m - (n_1 + n_2 + n_3) \notin S. \end{cases}$$

Moreover, $\widetilde{h}_1(\Delta_m) = 1$ and hence, in any minimal generating set of N_1 there is a unique element of degree m (S-graded Nakayama's Lemma).

Let $G = (g_1, g_2, g_3) \in N_1$ be a minimal generator of degree $m \in S$, where g_i has S-degree $m - \alpha_i n_i$, $i = 1, 2, 3$, and suppose that g_1 and g_2 are different from zero.

Since g_1 has S-degree $m - \alpha_1 n_1$, $m - (n_1 + n_2 + n_3) \notin S$ and $\alpha_1 n_1 = \gamma_{12} n_2 + \gamma_{13} n_3$ with $\gamma_{12},\ \gamma_{13} \neq 0$ we obtain

$$\begin{cases} m = \alpha_1 n_1 + \lambda n_2 \\ \text{or} \\ m = \alpha_1 n_1 + \mu n_3. \end{cases}$$

The integers λ and μ are positive because Δ_m is connected (equivalently $\widetilde{h}_0(\Delta_m) = 0$, see [3]). On the other hand, if $m = \alpha_1 n_1 + \lambda n_2$ and $\lambda \geq \alpha_2$, then, from the equality $\alpha_3 n_3 = \gamma_{31} n_1 + \gamma_{32} n_2$ with $\gamma_{31} < \alpha_1$ and $\gamma_{32} < \alpha_2$, one has $m - (n_1 + n_2 + n_3) = (\alpha_1 - 1)n_1 + (\lambda - 1)n_2 - n_3 \in S$, in contradiction with $m - (n_1 + n_2 + n_3) \notin S$. Therefore, we have that $0 < \lambda < \alpha_2$. Analogously, one can prove that $0 < \mu < \alpha_3$.

By the same arguments as above, using now that g_2 has S-degree $m - \alpha_2 n_2$ we obtain

$$\begin{cases} m = \alpha_2 n_2 + \lambda' n_1 \\ \text{or} \\ m = \alpha_2 n_2 + \mu' n_3, \end{cases}$$

with $0 < \lambda' < \alpha_1$ and $0 < \mu' < \alpha_3$.

The minimality of α_i, $i = 1, 2, 3$ implies that the only possibility (except permutation of $\{i, j, k\} = \{1, 2, 3\}$) is $m = \alpha_1 n_1 + \lambda n_2 = \alpha_2 n_2 + \mu' n_3$ with $0 < \lambda < \alpha_2$ and $0 < \mu' < \alpha_3$. So, the binomial $F = x_1^{\alpha_1} x_2^{\lambda} - x_2^{\alpha_2} x_3^{\mu'}$ lies in I_S, and the uniqueness of the integers γ_{ij}'s assures that $\lambda = \gamma_{32}$ and $\mu' = \gamma_{13}$. Furthermore, the only elements $m \in S$ such that $\widetilde{h}_1(\Delta_m) \neq 0$ are

$$m_1 = \alpha_1 n_1 + \gamma_{32} n_2 = \alpha_2 n_2 + \gamma_{13} n_3$$
$$m_2 = \alpha_1 n_1 + \gamma_{23} n_3 = \alpha_3 n_3 + \gamma_{12} n_2$$

(it is enough to check that the six possible cases are reduced to these ones).

Finally, notice that the column vectors of A_2 lie in N_1; indeed, $\Phi_1 \circ \Phi_2 = 0$ because $\alpha_k = \gamma_{jk} + \gamma_{ik}$ for any threesome $\{i, j, k\} = \{1, 2, 3\}$ (see Proposition 3.2 in [6]), and they are of degree m_1 and m_2, respectively. We conclude that N_1 is generated by the column vectors of A_2. □

Remark 2.4. There exist commutative algebra results (cf. [8]) which assure that a free resolution of I_S is $0 \longrightarrow k[\mathbf{x}]^2 \longrightarrow k[\mathbf{x}]^3 \longrightarrow k[\mathbf{x}] \longrightarrow k[S] \longrightarrow 0$. These arguments are used in [9] in order to get a similar explicit description of the minimal free resolution of I_S when S is not symmetric.

Corollary 2.5. *([7]) $k[S]$ is Gorenstein if and only if I_S is complete intersection (equivalently S is symmetric).*

Proof. It is enough to use that the Cohen-Macaulay type of $k[S]$ is the rank of the last free $k[\mathbf{x}]$-module in the minimal resolution. □

3 The hull resolution of a monomial curve in $\mathbb{A}^3(k)$.

Our aim in this section consist of characterize the hull resolution of I_S in terms of the semigroup S.

First of all we will study the structure of the hull complex $X = \text{hull}(M_{\mathcal{L}})$. Since $\mathcal{L} \cong \mathbb{Z}^2$ we will start with a characterization of all $\mathbb{Z}^2$-invariant triangulations of $\mathbb{R}^2$ whose set of vertices is $\mathbb{Z}^2$.

Given a finite set of vertices $\{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_r\}$, we write $\langle \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_r \rangle$ for

$$\left\{ \sum_{i=0}^{r} \lambda_i \mathbf{v}_i \in \mathbb{R}^n \ \mid \ \sum_{i=0}^{r} \lambda_i = 1 \text{ with } \lambda_i > 0 \right\}.$$

Definition 3.1. *Given a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{Z}^2$ we define the* simplicial complex associated with $\mathcal{B}$, *$K_{\mathcal{B}}$, to be an infinite simplicial complex such that*

(i) the set of vertices of $K_{\mathcal{B}}$ is $\mathbb{Z}^2$;

(ii) if $\Delta \in K_{\mathcal{B}}$, then $\Delta + \mathbf{b} \in K_{\mathcal{B}}$, for every $\mathbf{b} \in \mathbb{Z}^2$, that is, $K_{\mathcal{B}}$ is $\mathbb{Z}^2$-invariant;

(iii) $\langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle$ and $\langle 0, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1 \rangle$ are 2-simplices of $K_{\mathcal{B}}$.

Remark 3.2. In the light of definition above:

1. the simplicial complex $K_{\mathcal{B}}$ is unique;
2. the 2-simplices of $K_{\mathcal{B}}$ are $\langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle + \mathbf{b}$ and $\langle 0, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1 \rangle + \mathbf{b}$, for each $\mathbf{b} \in \mathbb{Z}^2$;
3. the geometric realization of $K_{\mathcal{B}}$ is $\mathbb{R}^2$, that is, $|K_{\mathcal{B}}| = \mathbb{R}^2$, for every basis $\mathcal{B}$ of $\mathbb{Z}^2$.

Proposition 3.3. *If K is a infinite simplicial complex such that*

(i) its set of vertices is $\mathbb{Z}^2$;

(ii) the geometric realization of K is $\mathbb{R}^2$;

(iii) if $\Delta \in K$, then $\Delta + \mathbf{b} \in K$, for every $\mathbf{b} \in \mathbb{Z}^2$,

then there exists a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{Z}^2$ such that $K = K_{\mathcal{B}}$.

Proof. By (i) we have that $0 \in \mathbb{Z}^2$ is a vertex of K, and by (ii) that there is a 2-simplex $\Delta \in K$ with $\Delta = \langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle$, for some $\mathbf{e}_1, \mathbf{e}_2 \in \mathbb{Z}^2$. It suffices to see that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis of $\mathbb{Z}^2$ to prove the result. It is clear that $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a basis of $\mathbb{R}^2$, otherwise, $\dim \Delta < 2$. On the other hand, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is not a basis of $\mathbb{Z}^2$, then there exists $\mathbf{b} \in \mathbb{Z}^2 \setminus \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2\}$ such that $\mathbf{b} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2$ with $0 < \alpha_i < 1,\ i = 1, 2$. Therefore, there is a face $F < \Delta$ with $\dim F > 0$ and $\langle \mathbf{b} \rangle \cap F \neq \emptyset$, in contradiction with (iii). From all this, taking $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$, it follows, by (iii), that K is the simplicial complex associated with $\mathcal{B}$. □

The results above assure that every infinite $\mathbb{Z}^2$-invariant triangulation K of $\mathbb{R}^2$ whose set of vertices is $\mathbb{Z}^2$ agrees with $K_{\mathcal{B}}$ for some basis $\mathcal{B}$ of $\mathbb{Z}^2$. Furthermore:

Corollary 3.4. *The only infinite $\mathcal{L}$-invariant triangulations of $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$ whose set of vertices is $\mathcal{L}$ are determined by a basis of $\mathcal{L}$. That is, any triangulation K of $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$ of this kind has got as facets $\langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle + \mathbf{b}$ and $\langle 0, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1 \rangle + \mathbf{b}$, for every $\mathbf{b} \in \mathcal{L}$ and some basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{L}$.*

Proof. Taking into account that there exist homeomorphisms from $\mathbb{R}^2$ to $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$ which send $\mathbb{Z}^2$ in $\mathcal{L}$, and therefore that transform $\mathbb{Z}^2$-invariance in $\mathcal{L}$-invariance, we are done. □

This last corollary is one of the key facts for a solution of our first problem. In our case the lattice module $M_{\mathcal{L}}$ is $k[\mathbf{x}]\{x_1^{v_1} x_2^{v_2} x_3^{v_3} \mid \sum_{i=1}^{3} v_i n_i = 0\}$. So the vertices of the polyhedron P_t are $\mathrm{t}^{\mathbf{v}} = (t^{v_1}, t^{v_2}, t^{v_3}) \in \mathbb{R}^3_+$ with $\sum_{i=1}^{3} v_i n_i = 0$ for t large enough.

Theorem 3.5. *There exists a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{L}$ such that the 2-cells of X consists of, one and only one, the following configurations:*

(1) squares $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle + \mathbf{b}$ such that $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2} \rangle + \mathbf{b}$ is not a 1-cell of X, for each $\mathbf{b} \in \mathcal{L}$;

(2) triangles $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2} \rangle + \mathbf{b}$ and $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle + \mathbf{b}$, for each $\mathbf{b} \in \mathcal{L}$.

Proof. First of all, since P_t is an 3-dimensional unbounded convex polyhedron (1.2) and X is the cell complex of its bounded faces, we have that the facets of X are 2-dimensional at most. Moreover, we know that the vertices of X are on $Y_1 := \{(x, y, z) \in \mathbb{R}^3_+ \mid x^{n_1} y^{n_2} z^{n_3} = 1\}$. So, we have that the facets of X are exactly of dimension two.

From all this it follows, since X is regular, that the geometric realization $|X|$ of X is homeomorphic to Y_1 (for a better understanding, consider, for instance, the homeomorphism which send each point $x \in |X|$ to the intersection of Y_1 with the line that contains x and $\mathbf{0} \in \mathbb{R}^3$). Therefore, every triangulation of $|X|$ is also a triangulation of Y_1.

Let us see now that there exists a $\mathcal{L}$-invariant triangulation of $|X| \cong Y_1$ whose set of vertices is $\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\}$. Let $\mathrm{Rep}(F_1), \ldots, \mathrm{Rep}(F_r)$ be the distinguished representatives of the orbits of $X/\mathcal{L}$ such that F_i is a 2-cell of X, $i = 1, \ldots, r$. For every $i = 1, \ldots, r$, we consider the triangulation of $|F_i|$ that is obtained after adding (if necessary) the 1-simplices $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{v}} \rangle$ for each $\mathrm{t}^{\mathbf{v}} \in |F_i|$, decomposing by this way $|F_i|$ in 2-simplices. Since X is $\mathcal{L}$-invariant, any other facet of X is $F_j + \mathbf{b}$ for some $j \in \{1, \ldots, r\}$ and $\mathbf{b} \in \mathcal{L}$. Thus, making a translation by $\mathcal{L}$ of these triangulations of $|F_i|, i = 1, \ldots, r$, to the remaining 2-cells, we obtain a infinite $\mathcal{L}$-invariant simplicial complex K whose set of vertices is $\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\}$ with $|K| = |X| \cong Y_1$, as desired.

On the other hand, the map $f : \mathcal{L} \otimes_{\mathbb{R}} \mathbb{Z} \longrightarrow Y_1$ such that $f(v_1, v_2, v_3) = (t^{v_1}, t^{v_2}, t^{v_3})$ is a homeomorphism of topological subspaces of $\mathbb{R}^3$ with the Euclidean topology which is also an isomorphism of $\mathbb{R}$-vector spaces which respects the action by $\mathcal{L}$. From where it is deduced that every $\mathcal{L}$-invariant triangulation of Y_1 whose set of vertices is $\{\mathrm{t}^{\mathbf{v}} \mid \mathbf{v} \in \mathcal{L}\}$ defines uniquely a $\mathcal{L}$-invariant triangulation of $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{R}$ whose set of vertices is $\mathcal{L}$, and vice versa. Therefore, by Corollary 3.4, we have that there exists a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathcal{L}$ such that the 2-simplices of K are $f(\langle 0, \mathbf{e}_1, \mathbf{e}_2 \rangle + \mathbf{b}) = \langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2} \rangle + \mathbf{b}$ and $f(\langle 0, \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_1 \rangle + \mathbf{b}) = \langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle + \mathbf{b}$, for each $\mathbf{b} \in \mathcal{L}$. From all this it immediately follows that representatives of the 2-cells of $X/\mathcal{L}$ are $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2} \rangle$ and $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle$ or $\langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_1}, \mathrm{t}^{\mathbf{e}_2}, \mathrm{t}^{\mathbf{e}_2 - \mathbf{e}_1} \rangle \supset \langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{e}_2} \rangle$. □

Once we have limited the suitable forms of the hull complex X, we can restrict the hull resolution of I_S to the two following cases.

Corollary 3.6. *The hull resolution of I_S admits exclusively two possibilities:*

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0$$

or

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Proof. By the results in section 1 and Theorem 3.5, we know that the number of 2-cell different modulo $\mathcal{L}$ and adjoint to $\mathrm{t}^{\mathbf{0}}$ are 1 or 2 and they are squares or triangles if it happens (1) or (2) in Theorem 3.5, respectively. □

In the view of result above, we have only to determine when it happens one or another resolution. This fact will only depend on the semigroup S. The non symmetric case can be reduced to well-known results.

Lemma 3.7. *If S is not symmetric, then the hull resolution of I_S is minimal. So the hull resolution is* $0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$

Proof. It is a well known fact (cf. [6]) that S is not symmetric if and only if the lattice ideal I_S is not complete intersection. In this case, by Theorem 2.2(b), I_S is also generic in the sense of [2], that is, there exists a system of generators of I_S of binomials with full support. Therefore the hull and minimal resolutions agree (cf. Example 3.12 in [2]). So, by Theorem 2.3(b), the hull resolution of I_S is $0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$ □

Assume now that I_S is complete intersection. By Theorem 2.2(a) we have that a minimal system of generators of I_S is $F_1 = x_i^{\alpha_i} - x_j^{\alpha_j}$ and $F_2 = x_k^{\alpha_k} - x_i^{\gamma_{ki}} x_j^{\gamma_{kj}}$ for some threesome $\{i, j, k\} = \{1, 2, 3\}$. Without loss of generality, we can suppose $i = 1,\ k = 2$ and $j = 3$, so $F_1 = x_1^{\alpha_1} - x_3^{\alpha_3}$ and $F_2 = x_2^{\alpha_2} - x_1^{\gamma_{21}} x_3^{\gamma_{23}}$.

Lemma 3.8. *If I_S is complete intersection, with the notation above, $\gamma_{21} = \gamma_{23}$ if and only if $\Gamma := \langle \mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{v}_1}, \mathrm{t}^{\mathbf{v}_2}, \mathrm{t}^{\mathbf{v}_1+\mathbf{v}_2} \rangle$ is a 2-cell of X, where $\mathbf{v}_1 = (\alpha_1, 0, -\alpha_3)$ and $\mathbf{v}_2 = (-\gamma_{21}, \alpha_2, -\gamma_{23})$.*

Proof. In first place, we suppose that Γ is a 2-cell of X. The points $\mathrm{t}^{\mathbf{0}},\ \mathrm{t}^{\mathbf{v}_1},\ \mathrm{t}^{\mathbf{v}_2}$ and $\mathrm{t}^{\mathbf{v}_1+\mathbf{v}_2}$ lie in a plane, equivalently, the determinant of the matrix

$$A(t) := \left(\mathrm{t}^{\mathbf{v}_1} - \mathrm{t}^{\mathbf{0}} | \mathrm{t}^{\mathbf{v}_2} - \mathrm{t}^{\mathbf{0}} | \mathrm{t}^{\mathbf{v}_1+\mathbf{v}_2} - \mathrm{t}^{\mathbf{0}}\right) \in \mathcal{M}_3(k(t))$$

has to be zero for t large enough, and this happens if and only if $\gamma_{21} = \gamma_{23}$.

Conversely, if $\gamma_{21} = \gamma_{23}$, then we have that $\det A(t) = 0$ which implies that Γ is a square. Thus, it suffices to see that Γ is a 2-cell of X. To do that, we will prove that every vertex in $X \setminus \Gamma$ is in one of the two half-spaces defined by the plane that contains the points $\mathrm{t}^{\mathbf{0}}, \mathrm{t}^{\mathbf{v}_1}, \mathrm{t}^{\mathbf{v}_2}$ and $\mathrm{t}^{\mathbf{v}_1+\mathbf{v}_2}$. An implicit equation of this plane is $ax + by + cz = d$ with

$$\begin{array}{rclcrcl} a &=& t^{\gamma_{21}}\left(t^{\alpha_2}-1\right)\left(t^{\alpha_3}-1\right); & \quad & c &=& t^{\alpha_3+\gamma_{21}}\left(t^{\alpha_2}-1\right)\left(t^{\alpha_1}-1\right); \\ b &=& \left(t^{\alpha_1+\alpha_3}-1\right)\left(t^{\gamma_{21}}-1\right); & & d &=& \left(t^{\alpha_2+\gamma_{21}}-1\right)\left(t^{\alpha_1+\alpha_3}-1\right). \end{array}$$

Note that a, b, c and d are strictly positive integers for t large enough. Moreover, since $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis $\mathcal{L}$, any other vertex in X is $\mathrm{t}^{a_1\mathbf{v}_1+a_2\mathbf{v}_2} = (t^{a_1\alpha_1 - a_2\gamma_{21}}, t^{a_2\alpha_2}, t^{-a_1\alpha_3 - a_2\gamma_{21}})$ for some a_1 and $a_2 \in \mathbb{Z}$. From both statements, it follows that if $a_1\alpha_1 - a_2\gamma_{21} \geq \alpha_1 + 1$ or $a_2\alpha_2 \geq \alpha_2 + 1$ or $-a_1\alpha_3 - a_2\gamma_{21} \geq 1$, then $\mathrm{t}^{a_1\mathbf{v}_1+a_2\mathbf{v}_2}$ is in the half-space $ax + by + cz > d$. Indeed, $ax = d$ if and only if

$$x = \frac{d}{a} = \frac{\left(t^{\alpha_2+\gamma_{21}}-1\right)\left(t^{\alpha_1+\alpha_3}-1\right)}{t^{\gamma_{21}}\left(t^{\alpha_2}-1\right)\left(t^{\alpha_3}-1\right)} =: f(t).$$

Since $\alpha_1 < \log_t f(t)$, $\lim_{t\to\infty} \log_t f(t) = \alpha_1$ and $f(t)$ is a strictly increasing function, we have that $x = t^{\alpha_1+\epsilon}$, with $0 < \epsilon < 1$, for t large enough. So, $a\, t^{\alpha_1+1} > d$ and consequently if $a_1\alpha_1 - a_2\gamma_{21} \geq \alpha_1 + 1$, then the point $\mathrm{t}^{a_1\mathbf{v}_1+a_2\mathbf{v}_2}$ is in $ax + by + cz > d$. The remaining inequalities are obtained by making a similar argument in each coordinate of $\mathrm{t}^{a_1\mathbf{v}_1+a_2\mathbf{v}_2}$.

Finally, the points of integer coordinates in $\mathbb{R}^2$ which lies in the region bounded by the half-spaces $\alpha_1 x - \gamma_{21} y \leq \alpha_1,\ y \leq 1$ and $\alpha_3 x + \gamma_{21} y \leq 0$ are $(0,0),\ (1,0),\ (0,1)$ and $(1,1)$. They precisely correspond to the points $\mathrm{t}^{\mathbf{0}},\ \mathrm{t}^{\mathbf{v}_1},\ \mathrm{t}^{\mathbf{v}_2}$ and $\mathrm{t}^{\mathbf{v}_1+\mathbf{v}_2}$ that already lie in $ax + by + cz = d$. Therefore, we have that all the vertices of $X \setminus \Gamma$ are in the half-space $ax + by + cz > d$. □

Lemma 3.9. *If I_S is complete intersection, with the notation above, $\gamma_{21} \neq \gamma_{23}$ (for every possible choice of them) if and only if the 2-cells of X are triangles.*

Proof. If the 2-cells of X are triangles, then $\langle t^{\mathbf{0}}, t^{\mathbf{v}_1}, t^{\mathbf{v}_2}, t^{\mathbf{v}_1+\mathbf{v}_2}\rangle$ is not a 2-cell of X, where $\mathbf{v}_1 = (\alpha_1, 0, -\alpha_3)$ and $\mathbf{v}_2 = (-\gamma_{21}, \alpha_2, -\gamma_{23})$, for any choice of γ_{21} and γ_{23}. From Lemma 3.8 it follows that $\gamma_{21} \neq \gamma_{23}$.

We now suppose that $\gamma_{21} \neq \gamma_{23}$ for every possible choice and that the 2-cells of X are not triangles. By Theorem 3.5 we have that the 2-cells of X are squares. Furthermore, there exists a basis $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ such that $\Delta := \langle t^{\mathbf{0}}, t^{\mathbf{e}_1}, t^{\mathbf{e}_2}, t^{\mathbf{e}_2-\mathbf{e}_1}\rangle$ is a 2-cell of X. Consequently the hull resolution of I_S is $0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0$.

On the other hand, since $\langle t^{\mathbf{0}}, t^{\mathbf{e}_2}\rangle$ is not a 1-cell of X, then the vertices of Δ adjoint to $t^{\mathbf{0}}$ are $t^{\mathbf{e}_1}$ and $t^{\mathbf{e}_2-\mathbf{e}_1}$. Necessarily, they have to be $\{\mathbf{e}_1, \mathbf{e}_2 - \mathbf{e}_1\} \subset \{\mathbf{u}, \mathbf{v}\}$, where $G_1 = \mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ and $G_2 = \mathbf{x}^{\mathbf{v}_+} - \mathbf{x}^{\mathbf{v}_-}$ is a minimal system of generators of I_S, otherwise, im $f_1 \neq \ker \Phi_0 = I_S$ and then the sequence $0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0$ could not be exact. From all this and by the uniqueness in Theorem 2.2(b), it follows $\{\mathbf{u}, \mathbf{v}\} = \{\pm\mathbf{v}_1, \pm\mathbf{v}_2\}$, for some $\mathbf{v}_1 = (\alpha_1, 0, -\alpha_3)$ and $\mathbf{v}_2 = (-\gamma_{21}, \alpha_2, -\gamma_{23})$. Then:

- $\mathbf{e}_1 = \pm\mathbf{v}_i, \mathbf{e}_2 - \mathbf{e}_1 = \pm\mathbf{v}_j$ and consequently $\mathbf{e}_2 = \pm(\mathbf{v}_i + \mathbf{v}_j)$, with $\{i, j\} = \{1, 2\}$.
- $\mathbf{e}_1 = \mp\mathbf{v}_i, \mathbf{e}_2 - \mathbf{e}_1 = \pm\mathbf{v}_j$ and consequently $\mathbf{e}_2 = \pm\mathbf{v}_j \mp \mathbf{v}_i$, with $\{i, j\} = \{1, 2\}$.

Any of these eight possibilities implies that $\Gamma := \langle t^{\mathbf{0}}, t^{\mathbf{v}_1}, t^{\mathbf{v}_2}, t^{\mathbf{v}_1+\mathbf{v}_2}\rangle$ is a 2-cell of X, because $\Gamma = \Delta + \mathbf{b}$ for some $\mathbf{b} \in \mathcal{L}$, in contradiction with Lemma 3.8. □

All these results are the proof of our main theorem.

Theorem 3.10. *Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbb{Z}_+$, $i = 1, 2, 3$, and $\gcd(n_1, n_2, n_3) = 1$. The hull resolution of I_S is*

$$0 \longrightarrow k[\mathbf{x}] \xrightarrow{f_2} k[\mathbf{x}]^2 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0,$$

when S is symmetric with $\alpha_i n_i = \alpha_j n_j$ and there exist $\gamma_{ki} = \gamma_{kj}$ such that $\alpha_k n_k = \gamma_{ki} n_i + \gamma_{kj} n_j$, for some threesome $\{i, j, k\} = \{1, 2, 3\}$. Otherwise the hull resolution of I_S is

$$0 \longrightarrow k[\mathbf{x}]^2 \xrightarrow{f_2} k[\mathbf{x}]^3 \xrightarrow{f_1} k[\mathbf{x}] \xrightarrow{\Phi_0} k[S] \longrightarrow 0.$$

Corollary 3.11. *Let S be a semigroup of positive integers generated by $\{n_1, n_2, n_3\}$, with $n_i \in \mathbb{Z}_+$, $i = 1, 2, 3$, and $\gcd(n_1, n_2, n_3) = 1$. The hull resolution of I_S is minimal if and only if*

- *S is not symmetric, or*
- *S is symmetric with $\alpha_i n_i = \alpha_j n_j$ and there exist $\gamma_{ki} = \gamma_{kj}$ such that $\alpha_k n_k = \gamma_{ki} n_i + \gamma_{kj} n_j$, for some threesome $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. It follows from Theorem 3.10 and Theorem 2.3. □

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