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**On Kummer extensions of the power series field**

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## Abstract

In this paper we study the Kummer extensions of a power series field  $K = k((X_1, \dots, X_r))$ , where  $k$  is an algebraically closed field of arbitrary characteristic.

## 1 Terminology and notation

Let  $k$  be an algebraically closed field,  $X_1, \dots, X_r$  indeterminates formally independent over  $k$ , and let  $K$  and  $L_m$  be the fields

$$K = k((X_1, \dots, X_r)), L_m = k\left(\left(X_1^{1/m}, \dots, X_r^{1/m}\right)\right),$$

where  $m$  is a non negative integer, not divisible by the characteristic of  $k$ .

The extension  $K \subset L_m$  is trivially normal, finite and separable, its Galois group being  $G \simeq (C_m)^r$ , where  $C_m$  stands for the cyclic group of  $m$  elements. The elements of  $G$  will be noted

$$\begin{aligned} (a_1, \dots, a_r) : L_m &\longrightarrow L_m, & 0 \leq a_i < m \\ X_l &\longmapsto \omega^{a_l} X_l \end{aligned}$$

where  $\omega \in k$  is an  $m$ -th primitive root of the unity.

Let  $R$  and  $S_m$  be the rings

$$R = k[[X_1, \dots, X_r]], \quad S_m = k\left[\left[X_1^{1/m}, \dots, X_r^{1/m}\right]\right].$$

The elements of  $S_m$  will be called *Puiseux power series*.

Our field of study will be Kummer extensions of  $K$ . In order to do that, recall ([1]) that a *Kummer extension* of exponent  $n$  of a field  $F$  (which must contain a primitive  $n$ -th root of the unity and hence its characteristic cannot divide  $n$ ), is the splitting field of a polynomial

$$(Z^n - \alpha_1) \dots (Z^n - \alpha_q), \quad \text{with } \alpha_1, \dots, \alpha_q \in F.$$

We will first explore the Kummer extensions which are subextensions of  $K \subset L_m$ . It is obvious that all separable extensions of  $K$  generated by monomials lying in some  $S_m$  are Kummer extensions. On the other hand, notice that an extension generated by monomials in  $S_m$  should contain a Puiseux power series which generates it (using the primitive element theorem, as  $k$  must be infinite). In the next section, we will prove the converse, hence proving that all subextensions of  $K \subset L_m$  are Kummer. This will be done in two different ways. First we will show a straightforward proof and, subsequently a more involved process, which will carry more information and which is, in an ample sense, constructive.

It should be noted that the results in this section also admit a Galois theory approach, as shown in [6]. The connection of these results with the theory of quasi-ordinary surface singularities, sketched below, becomes clear after [5] and [7], where the semigroup point of view is exploited in order to obtain geometric information on these singularities.

In the last section, we will make some general remarks about Kummer extensions not contained in any  $L_m$ , proving that all such extensions are contained in a field

$$K\left(\left(X_1^{\alpha_1/m}\right)\right) \dots \left(\left(X_r^{\alpha_r/m}\right)\right),$$

for some  $\alpha_1, \dots, \alpha_r, m \in \mathbf{N}$ .

Most of the arguments given here (notably all from section 2) can be completely translated word-by-word to the analytic context. We hope that this work will be useful as a step to understand the geometry of algebroid (analytic) hypersurfaces which admit a Puiseux-like parametrization. In fact, distinguished exponents have proved to be a useful tool for the surface case (characteristic 0), as shown in [8]. This results have led us to expect that some deeper application of class field theory tools may help to the study of the geometry and the topology of these varieties.

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## 2 Distinguished exponents of a Puiseux power series

If  $\zeta \in S_m$  is written as

$$\zeta = \sum c_{i_1 \dots i_r} X_1^{i_1/m} \dots X_r^{i_r/m}, \quad c_{i_1 \dots i_r} \in k,$$

then the set

$$\Delta(\zeta) = \{(i_1, \dots, i_r) \mid c_{i_1 \dots i_r} \neq 0\} \subset \mathbf{N}^r$$

will be called (a bit carelessly) the *set of exponents* of  $\zeta$ .

**Definition.**– Given  $\zeta \in S_m$ , a finite subset

$$\left\{ \left( i_1^{(1)}, \dots, i_r^{(1)} \right), \dots, \left( i_1^{(s)}, \dots, i_r^{(s)} \right) \right\} \subset \Delta(\zeta)$$

will be called a *set of distinguished exponents* of  $\zeta$  if

$$K(\zeta) = K \left( X_1^{i_1^{(1)}/m} \dots X_r^{i_r^{(1)}/m}, \dots, X_1^{i_1^{(s)}/m} \dots X_r^{i_r^{(s)}/m} \right).$$

In order to prove that all separable extensions of  $K$  generated by a Puiseux power series are Kummer extensions, it suffices to check that all Puiseux power series in  $S_m$ , where  $m$  is not divisible by  $\text{ch}(k)$ , possess a set of distinguished exponents.

**Proposition.**– Let  $\zeta \in S_m$ . For all  $(a_1, \dots, a_r) \in G$  let us write

$$\Delta_{(a_1, \dots, a_r)}(\zeta) = \min_{<} \{ \Delta(\zeta - (a_1, \dots, a_r)(\zeta)) \},$$

where  $<$  is the partial product order. Note there is only a finite number of  $>$ -minimal elements in  $\Delta(\zeta - (a_1, \dots, a_r)(\zeta))$ . Then the set

$$P = \bigcup_{(a_1, \dots, a_r) \in G} \Delta_{(a_1, \dots, a_r)}(\zeta)$$

is a set of distinguished exponents of  $\zeta$ .

**Proof.**– In the situation above, if

$$P = \left\{ \left( i_1^{(1)}, \dots, i_r^{(1)} \right), \dots, \left( i_1^{(t)}, \dots, i_r^{(t)} \right) \right\},$$

we will write for short

$$K[P] = K \left[ X_1^{i_1^{(l)}} \dots X_r^{i_r^{(l)}} \mid l = 1, \dots, t \right].$$

It is clear that  $K[P] \subset K[\zeta]$ , as every element of  $G$  leaving  $\zeta$  fixed, does so with the monomials having exponents in  $P$ . On the other hand, if we take

$$(a_1, \dots, a_r) \in \text{Gal}(K[P]/K) \setminus \text{Gal}(K[\zeta]/K),$$

it must hold

$$\Delta(\zeta - (a_1, \dots, a_r)(\zeta)) \subset \Delta(\zeta) \setminus P,$$

which is a contradiction by the definition of  $P$ . This proves the result.

Note that the above result, although outputs a set of distinguished exponents, says little or nothing at all about the relations among the exponents in  $\Delta(\zeta)$  or the degree of the extension  $K \subset K[\zeta]$ .

In order to obtain more information we will describe in what follows a different and much more versatile process for obtaining such a set for a given series  $\zeta \in S_m$ . First of all we fix a total ordering in  $\mathbf{N}^r$ , say  $\prec$ , and assume that  $m$  is the minimal denominator for  $\zeta$  (that is,  $\zeta \notin S_q$  for all  $q < m$ ).

For a given matrix  $A$  of  $t$  rows and  $u$  columns, whose elements are integers, we will write

$$(l) \text{gcd}(A) = \text{gcd}(\text{minors of order } l \text{ in } A),$$

for all  $l = 1, \dots, \min\{t, u\}$ .

**Step 1.**– Consider the  $r \times r$  matrix

$$M_0 = \begin{pmatrix} m & 0 & \dots & 0 \\ 0 & m & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & m \end{pmatrix},$$

which obviously verifies  $(r) \text{gcd}(M_0) = m^r$ .

**Step 2.**– Define the sets  $\Delta_0 = \Delta(\zeta)$  and

$$\Delta'_0 = \left\{ (i_1, \dots, i_r) \in \Delta_0 \mid (r) \text{gcd}(M_0) = (r) \text{gcd} \left( M_0 \begin{pmatrix} i_1 \\ \vdots \\ i_r \end{pmatrix} \right) \right\}.$$

(These exponents are trivially those representing monomials of  $\zeta$  which lie in  $R$ ).

**Step 3.**– Write  $\Delta_1 = \Delta_0 \setminus \Delta'_0$ , define the first distinguished pair by

$$\left( i_1^{(1)}, \dots, i_r^{(1)} \right) = \min_{<} (\Delta_1);$$

and consider the matrix

$$M_1 = \left( M_0 \left| \begin{array}{c} i_1^{(1)} \\ \vdots \\ i_r^{(1)} \end{array} \right. \right).$$

**Step 4.**— Once the distinguished pairs

$$\left( i_1^{(1)}, \dots, i_r^{(1)} \right), \dots, \left( i_1^{(l)}, \dots, i_r^{(l)} \right),$$

the set  $\Delta_l$  and the matrix  $M_l$  are defined, consider

$$\Delta'_l = \left\{ (i_1, \dots, i_r) \in \Delta_l \left| (r) \gcd(M_l) = (r) \gcd \left( M_l \left| \begin{array}{c} i_1 \\ \vdots \\ i_r \end{array} \right. \right) \right. \right\}.$$

**Step 5.**— Write  $\Delta_{l+1} = \Delta_l \setminus \Delta'_l$ , define the  $(l+1)$ -th distinguished pair by

$$\left( i_1^{(l+1)}, \dots, i_r^{(l+1)} \right) = \min_{\prec} (\Delta_{l+1});$$

and consider the matrix

$$M_{l+1} = \left( M_l \left| \begin{array}{c} i_1^{(l+1)} \\ \vdots \\ i_r^{(l+1)} \end{array} \right. \right).$$

**Remark.**— The previous procedure must give a finite number of distinguished pairs, as for every  $l > 0$  we have

$$(r) \gcd(M_{l-1}) > (r) \gcd(M_l),$$

so we must end up with a finite set

$$P = \left\{ \left( i_1^{(1)}, \dots, i_r^{(1)} \right), \dots, \left( i_1^{(s)}, \dots, i_r^{(s)} \right) \right\}.$$

Now, as above,  $K[P] \subset K[\zeta]$ , keeping our previous notation. So, for proving that  $P$  is a set of distinguished monomials, it suffices proving the following result:

**Proposition.**— Let there be

$$P_1 = \left\{ X_1^{j_1^{(1)}/m} \dots X_r^{j_r^{(1)}/m}, \dots, X_1^{j_1^{(t)}/m} \dots X_r^{j_r^{(t)}/m} \right\},$$

$$P_2 = P_1 \cup \left\{ X_1^{j_1^{(t+1)}/m} \dots X_r^{j_r^{(t+1)}/m} \right\}$$

two sets of monomials in  $S_m$  (not in any  $S_q$ , with  $q < m$ ), such that

$$(r) \gcd(M_1) = (r) \gcd(M_2),$$

where

$$M_1 = \begin{pmatrix} m & \dots & 0 & j_1^{(1)} & \dots & j_1^{(t)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & m & j_r^{(1)} & \dots & j_r^{(t)} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} m & \dots & 0 & j_1^{(1)} & \dots & j_1^{(t)} & j_1^{(t+1)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \dots & m & j_r^{(1)} & \dots & j_r^{(t)} & j_r^{(t+1)} \end{pmatrix}.$$

Then  $K[P_1] = K[P_2]$ .

**Proof.**— The point is proving  $K[P_1] \supset K[P_2]$  and, for this, it is necessary and sufficient showing that, if we call  $G_k = \text{Gal}(L_m/K[P_k])$ , for  $k = 1, 2$ ; then  $G_1 = G_2$ .

Define the set

$$H_1 = \left\{ (i_1, \dots, i_r) \in (\mathbf{Z}/\mathbf{Z}m)^r \left| X_1^{i_1/m} \dots X_r^{i_r/m} \in K[P_1] \right. \right\}.$$

So,  $H_1$  contains, up to multiples of  $m$  in all coordinates, those monomials which remain fixed by the elements of  $G_1$ . Writing up these elements in the form  $(a_1, \dots, a_r)$  it means that

$$(i_1, \dots, i_r) \in H_1 \iff \sum_{l=1}^r a_l i_l = 0 \pmod{m}, \forall (a_1, \dots, a_r) \in G_1,$$

and also, in particular,

$$H_1 = \left\langle \left( j_1^{(1)}, \dots, j_r^{(1)} \right), \dots, \left( j_1^{(t)}, \dots, j_r^{(t)} \right) \right\rangle.$$

Therefore  $H_1$  is clearly a subgroup of  $G$  (non-canonically identified with  $(\mathbf{Z}/\mathbf{Z}m)^r$ ), but it also admits another interpretation. In fact,

$$H_1 \simeq \text{Hom}(G/G_1, \mathbf{Z}/\mathbf{Z}m),$$

identifying  $(i_1, \dots, i_r) \in H_1$  with

$$\begin{aligned} f_{(i_1, \dots, i_r)} : G/G_1 &\longrightarrow \mathbf{Z}/\mathbf{Z}m \\ (x_1, \dots, x_r) + G_1 &\longmapsto \sum_{l=1}^r x_l i_l \end{aligned}$$

As  $G$  is the direct sum of  $r$  cyclic groups of order  $m$ , we have that  $G/G_1$  can be written up as

$$G/G_1 = C_{a_1} \oplus \dots \oplus C_{a_c}, \text{ where } a_l | m, \forall l = 1, \dots, c.$$

This leads to

$$H_1 \simeq \text{Hom}(G/G_1, \mathbf{Z}/\mathbf{Z}m) \simeq \bigoplus_{l=1}^c \text{Hom}(C_{a_l}, \mathbf{Z}/\mathbf{Z}m) \simeq \bigoplus_{l=1}^c C_{a_l} \simeq G/G_1,$$

as  $a_l | m$ , for all  $l$ .

On the other hand  $|G/G_1|$  (that is,  $[K[P_1] : K]$ ), is precisely  $|H_1|$  and hence,

$$|G_1| = |G/H_1|.$$

Let us calculate  $|G/H_1|$ . First of all, instead of writing the group as

$$(\mathbf{Z}/\mathbf{Z}m)^r / \left\langle \left( j_1^{(1)}, \dots, j_r^{(1)} \right), \dots, \left( j_1^{(t)}, \dots, j_r^{(t)} \right) \right\rangle,$$

we will do it as  $\mathbf{Z}^r / \widehat{H}_1$ , where

$$\widehat{H}_1 = \left\langle (m, 0, \dots, 0), \dots, (0, 0, \dots, m), \left( j_1^{(1)}, \dots, j_r^{(1)} \right), \dots, \left( j_1^{(t)}, \dots, j_r^{(t)} \right) \right\rangle.$$

Let us write  $\varphi$  a generic element of  $\text{Hom}(\mathbf{Z}^r, \mathbf{Q}/\mathbf{Z})$  with  $\widehat{H}_1 \subset \ker(\varphi)$ , and  $\tilde{\varphi}$  its factorization through  $\mathbf{Z}^r / \widehat{H}_1$ . According to [2], prop. 8;

$$\mathbf{Z}^r / \widehat{H}_1 \simeq \text{Hom}\left(\mathbf{Z}^r / \widehat{H}_1, \mathbf{Q}/\mathbf{Z}\right),$$

and each of these morphisms is characterized by the images of the canonical generating set of  $\mathbf{Z}^r / \widehat{H}_1$ , say

$$\alpha_l = \tilde{\varphi}\left(e_l + \widehat{H}_1\right),$$

where  $e_l$  stands for the  $l$ -th element of the canonical basis of  $\mathbf{Z}^r$ .

But  $\widehat{H}_1 \subset \ker(\varphi)$  is equivalent to

$$(\alpha_1 \dots \alpha_r) \begin{pmatrix} m & \dots & 0 & j_1^{(1)} & \dots & j_1^{(t)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & m & j_r^{(1)} & \dots & j_r^{(t)} \end{pmatrix} = (0 \dots 0).$$

Again by [2], cor. 1, we can find some linear forms  $L_1, \dots, L_r$  with coefficients on  $\mathbf{Z}$  such that the previous relations are equivalent to

$$(L_1(\alpha_1, \dots, \alpha_r) \dots L_r(\alpha_1, \dots, \alpha_r)) \begin{pmatrix} \eta_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & \eta_r & 0 & \dots & 0 \end{pmatrix} = (0 \dots 0),$$

where  $\eta_1 = (1) \text{gcd}(M_1) = 1$  and  $\eta_1 \dots \eta_l = (l) \text{gcd}(M_1)$ . Therefore, as this equality must hold in  $\mathbf{Q}/\mathbf{Z}$ , it is plain that there are exactly  $(r) \text{gcd}(M_1)$  different morphisms in  $\text{Hom}\left(\mathbf{Z}^r / \widehat{H}_1, \mathbf{Q}/\mathbf{Z}\right)$ , hence

$$|G_1| = \frac{m^r}{|H_1|} = \frac{m^r}{m^r / (r) \text{gcd}(M_1)} = (r) \text{gcd}(M_1).$$

Doing exactly the same with  $G_2$  we find

$$|G_2| = (r) \text{gcd}(M_2) = (r) \text{gcd}(M_1) = |G_1|.$$

This finishes the proof, as  $G_1 \subset G_2$ .

**Corollary.**– If  $\zeta \in S_m$ , having a set of distinguished exponents

$$P = \left\{ \left( i_1^{(1)}, \dots, i_r^{(1)} \right), \dots, \left( i_1^{(s)}, \dots, i_r^{(s)} \right) \right\} \subset \Delta(\zeta),$$

then

$$(a) \Delta(\zeta) \subset \sum_{l=i}^s \mathbf{Z} \left( i_1^{(l)}, \dots, i_r^{(l)} \right), \quad \text{mod } \mathbf{Z}m \times \dots \times \mathbf{Z}m.$$

$$(b) [K(\zeta) : K] = m^r / (r) \gcd(M), \text{ where}$$

$$M = \begin{pmatrix} m & \dots & 0 & i_1^{(1)} & \dots & i_1^{(s)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \dots & m & i_r^{(1)} & \dots & i_r^{(s)} \end{pmatrix}.$$

Let us remember that a power series  $\zeta \in S_m$  is called quasi-ordinary ([4]) if, for all  $(a_1, \dots, a_r) \in G$ , it holds

$$\zeta - (a_1, \dots, a_r)(\zeta) = M_{(a_1, \dots, a_r)} u_{(a_1, \dots, a_r)} \left( X_1^{1/m}, \dots, X_r^{1/m} \right),$$

where  $M_{(a_1, \dots, a_r)}$  is a monomial in  $S_m$  and  $u_{(a_1, \dots, a_r)}$  is a unit.

**Corollary.**– Let  $K \subset K'$  be a separable field extension. The following facts are equivalent:

$$(a) K' \subset L_m.$$

$$(b) K' \text{ can be generated by a finite set of monomials in } S_m.$$

$$(c) K' \text{ can be generated by a quasi-ordinary power series.}$$

**Proof.**– We have just proved (a)  $\iff$  (b) and, as (c)  $\implies$  (a) is trivial, we are finished if we prove (b)  $\implies$  (c). But, for a given set of monomials in  $S_m$ , say  $\{M_1, \dots, M_t\}$ , note that

$$K[M_1, \dots, M_t] = K[M_1, M_1 M_2, \dots, M_1 M_2 \dots M_t].$$

Now, if we apply the primitive element theorem to the second set of generators we can find  $\alpha_1, \dots, \alpha_t \in k$  such that

$$K[M_1, \dots, M_t] = K[\alpha_1 M_1 + \alpha_2 M_1 M_2 + \dots + \alpha_t M_1 M_2 \dots M_t],$$

and it is plain that  $\alpha_1 M_1 + \dots + \alpha_t M_1 M_2 \dots M_t$  is quasi-ordinary.

**Remark.**– This process enables to compute (on equal footing) two well-known (sets of) arithmetic data which are most useful in algebraic geometry, as are the Puiseux pairs of a plane curve ([9]) and the characteristic pairs of a quasi-ordinary surface ([4]) (both of them for  $k = \mathbf{C}$ ).

**Example.**– Let us do the Puiseux pairs case: Assume we have a plane algebroid curve given by  $f(X, Y) \in \mathbf{C}[[X, Y]]$  and a Puiseux branch, which can always be represented (up to a change of variables) as

$$\begin{aligned} Y = \zeta \left( X^{1/m} \right) &= c_{\beta_1} X^{\beta_1/m} + \sum_{l=1}^{h_1} c_{\beta_1 + l e_1} X^{(\beta_1 + l e_1)/m} + \dots \\ &\dots + c_{\beta_g} X^{\beta_g/m} + \sum_{l=1}^{\infty} c_{\beta_g + l e_g} X^{(\beta_g + l e_g)/m}, \end{aligned}$$

where we can assume  $m < \beta_1 < \dots < \beta_g$ ,  $\beta_k \notin \mathbf{Z}m$  for all  $k = 1, \dots, g$  and, in addition, if we call

$$\beta_1 = p_1 e_1, \quad m = q_1 e_1, \quad \gcd(p_1, q_1) = 1$$

$$e_{l-1} = q_l e_l, \quad \beta_l = p_l e_l, \quad \gcd(p_l, q_l) = 1; \quad \forall l = 2, \dots, g,$$

then the pairs  $(p_1, q_1), \dots, (p_g, q_g)$  are called the *Puiseux pairs* of the curve. Note that these pairs are determined (and they determine as well) by the set

$$\{m, \beta_1, \dots, \beta_g\},$$

called by Zariski the *characteristic of the branch*  $\zeta$ . Also is direct from the formulae above that

$$e_1 = \gcd(m, \beta_1), \quad e_l = \gcd(e_{l-1}, \beta_l), \quad \forall l = 2, \dots, g.$$

If we apply our process to the set of exponents on  $\Delta(\zeta)$  using, for instance, the natural ordering on  $\mathbf{N}$ , we start up with

$$M_0 = (m)$$

and then choose the smaller element on  $\Delta$ , that is,  $\beta_1$ , which, by the above conditions, happens to verify  $\gcd(m, \beta_1) < m$ , so  $i^{(1)} = \beta_1$ .

Assume we have already computed the first  $l$  distinguished exponents, which coincide with  $\beta_1, \dots, \beta_l$  (necessarily in this order because of our choosing of the ordering on  $\mathbf{Z}$ ). Then we have the matrix

$$M_l = (m \ \beta_1 \ \dots \ \beta_l),$$

and  $\gcd(m, \beta_1, \dots, \beta_l) = e_l$ , by the above considerations. We have discarded in previous steps those monomials which can be written as a combination of some  $\beta_t$  and  $e_t$ , for  $t < l$ . In the same way, then, we discard now those elements in  $\Delta$  which do not make smaller the previous gcd, which are, precisely, those which can be written up as a combination of  $\beta_l$  and  $e_l$ .

By definition of  $\beta_{l+1}$ , it has to be the minimal element not yet discarded, and this proves that our procedure must end up computing the set  $\{\beta_1, \dots, \beta_g\}$ .

**Example.**— Consider now a quasi-ordinary surface. For the definitions and main properties of so-called the characteristic monomials of a quasi-ordinary branch  $\zeta$  we refer to [4] or [3]. For a quasi-ordinary branch

$$\zeta = \sum_{i,j} c_{ij} X^{i/m} Y^{j/m}$$

parametrizing a quasi-ordinary algebroid surface, the characteristic monomials  $\{(i_1, j_1), \dots, (i_s, j_s)\} \subset \mathbf{N}^2$  are determined (up to a technical process called normalization) by the following facts:

- (1) They are totally ordered by the partial product order  $<$

$$(i_1, j_1) < (i_2, j_2) < \dots < (i_s, j_s).$$

- (2) They are a set of distinguished exponents.

- (3) For any  $1 \leq t \leq s$ ,  $X^{i_t/m} Y^{j_t/m}$  lies in  $K((X^{i_1/m} Y^{j_1/m}, \dots, X^{i_t/m} Y^{j_t/m}))$  if and only if

$$(i, j) \in \sum_{l=1}^t \mathbf{Z}(i_l, j_l) \pmod{\mathbf{Z}m \times \mathbf{Z}m}.$$

- (4) For any  $1 \leq t \leq s$ ,

$$X^{i_t/m} Y^{j_t/m} \notin K((X^{i_1/m} Y^{j_1/m}, \dots, X^{i_{t-1}/m} Y^{j_{t-1}/m})).$$

- (5) If  $c_{ij} \neq 0$  then

$$(i, j) \in \sum_{l=1}^s \mathbf{Z}(i_l, j_l) \pmod{\mathbf{Z}m \times \mathbf{Z}m}.$$

So, if we take  $\zeta$  and apply our constructive process fixing a *graded* total ordering  $\prec$ , it is clear from above that  $(i_1, j_1)$  must be the minimal element of  $\Delta(\zeta)$  for  $\prec$ . Now, from (3) and (4) the second element chosen in the process must be precisely  $(i_2, j_2)$ . The argument goes on for further exponents and then (2) assures us that, once  $\{(i_1, j_1), \dots, (i_s, j_s)\}$  have been extracted, the process is finished.

Note that the choice of a graded ordering is not necessary in order to obtain the characteristic exponents (many non-graded orders might work as well).

### 3 General Kummer extensions

We will make now some remarks about Kummer extensions of  $K$  of any kind. In order to do that observe that we can reduce the problem to that of the splitting field of a polynomial  $F(Z) = Z^n - \zeta$ , where  $\zeta \in R$ .

**Remark.**— First of all, mind that, if  $\zeta$  is an irreducible power series not associated with any of the  $X_i$ , we cannot hope the splitting field of  $F$  to be a subextension of some  $L_m$ . In fact,  $\zeta$  defines a valuation  $v_\zeta$  of  $K$  that is unramified over  $L_m$ , as  $v_\zeta(X_i) = 0$ , for all  $i$ , but it is obviously ramified over the splitting field. However, we can prove a resembling result.

**Proposition.**— In the above situation, there are  $\alpha_1, \dots, \alpha_r \in \mathbf{N}$  such that the splitting field of  $F$  is a subextension of

$$K \left( \left( X_1^{\alpha_1/n} \right) \dots \left( X_r^{\alpha_r/n} \right) \right).$$

**Proof.**— We will do the proof by induction on  $r$ , being the case  $r = 1$  direct from the Newton–Puiseux theorem. So assume that, for all  $\eta \in R$ , there exists a set of positive integers  $\{\alpha_1, \dots, \alpha_r\}$  such that

$$K[\sqrt[n]{\eta}] = K \left( \left( X_1^{\alpha_1/n} \right) \dots \left( X_r^{\alpha_r/n} \right) \right);$$

and fix a power series  $\zeta \in R[[X_{r+1}]]$ , with  $\nu(\zeta) = \lambda_0 \geq 0$ , where  $\nu$  is the usual order with respect to  $X_{r+1}$ . We want to find a root of  $Z^n - \zeta \in K'[Z]$ , where  $K' = k((X_1, \dots, X_{r+1}))$ .

Let us call from now on  $\zeta_i$  the approximate  $n$ -th root of  $\zeta$  (up to order  $i$  in  $X_{r+1}$ ), which will be constructed in what follows. The term with minimal degree on  $X_{r+1}$  of a  $\sqrt[n]{\zeta}$ , must be of the form  $c_{\lambda_0/n} X_{r+1}^{\lambda_0/n}$ , where it must hold

$$c_{\lambda_0/n}^n = a_{\lambda_0},$$

So there must be a set of monomials  $\left\{ X_1^{\alpha_1/n}, \dots, X_r^{\alpha_r/n} \right\} \subset S_n$  such that

$$K' \left[ \sqrt[n]{a_{\lambda_0}} \right] = K' \left[ c_{\lambda_0/n} \right] \subset K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right).$$

Hence we can write  $\zeta_{\lambda_0} = c_{\lambda_0/n} X_{r+1}^{\lambda_0/n}$ , which verifies

$$\zeta_{\lambda_0} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right) \left[ \left[ X_{r+1}^{\lambda_0/n} \right] \right], \quad \nu(\zeta_{\lambda_0}^n - \zeta) = \lambda_1 > \lambda_0.$$

Now, in the same way, the following term with minimal degree on  $X_{r+1}$  of any  $\sqrt[n]{\zeta}$ , must be of the form  $c_{\lambda_1 - [\lambda_0(n-1)/n]} X_{r+1}^{\lambda_1 - [\lambda_0(n-1)/n]}$ , where it must hold now

$$n c_{\lambda_1 - [\lambda_0(n-1)/n]} c_{\lambda_0/n}^{(n-1)/n} = a_{\lambda_1},$$

with  $a_{\lambda_1}$  the initial form (with respect to  $X_{r+1}$ ) of  $\zeta_{\lambda_0}^n - \zeta$  and hence

$$c_{\lambda_1 - [\lambda_0(n-1)/n]} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right).$$

We write  $\zeta_{\lambda_1} = c_{\lambda_0/n} X_{r+1}^{\lambda_0/n} + c_{\lambda_1 - [\lambda_0(n-1)/n]} X_{r+1}^{\lambda_1 - [\lambda_0(n-1)/n]}$  and, as above, it holds

$$\zeta_{\lambda_1} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right) \left[ \left[ X_{r+1}^{\lambda_0/n} \right] \right], \quad \nu(\zeta_{\lambda_1}^n - \zeta) = \lambda_2 > \lambda_1.$$

Note that, though  $\zeta_{\lambda_1}$  is a Puiseux power series, all the exponents in  $X_{r+1}$  of  $\zeta_{\lambda_1}^n$  are positive integers: in fact, they are  $\{\lambda_0, \lambda_1, 2\lambda_1 - \lambda_0, \dots, n\lambda_1 - (n-1)\lambda_0\}$ . In particular, this shows  $\lambda_2 \in \mathbf{N}$ .

Assume now we have constructed  $\zeta_{\lambda_s}$ , verifying

- $\zeta_{\lambda_s} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right) \left[ \left[ X_{r+1}^{\lambda_0/n} \right] \right]$ ,
- $\nu(\zeta_{\lambda_s}^n - \zeta) = \lambda_{s+1} > \lambda_s > \dots > \lambda_0$ , with all  $\lambda_i \in \mathbf{N}$ .
- The series  $\zeta_{\lambda_s}$  has the form

$$\zeta_{\lambda_s} = \sum_{j=0}^s c_{L_j(\lambda_1, \dots, \lambda_j) - \beta_j \lambda_0 + \lambda_0/n} X_{r+1}^{L_j(\lambda_1, \dots, \lambda_j) - \beta_j \lambda_0 + \lambda_0/n},$$

where  $L_j$  is a linear form with positive coefficients and  $\beta_j$  a positive integer, for  $j = 1, \dots, s$ ; being  $L_0 = \alpha_0 = 0$ .

If we call  $a_{\lambda_{s+1}}$  the initial form of  $\zeta_{\lambda_s}^n - \zeta$  then it is clear that we have to define

$$\zeta_{\lambda_{s+1}} = \zeta_{\lambda_s} + c_{\lambda_{s+1} - [\lambda_0(n-1)/n]} X_{r+1}^{\lambda_{s+1} - [\lambda_0(n-1)/n]},$$

where it must hold

$$n c_{\lambda_{s+1} - [\lambda_0(n-1)/n]} c_{\lambda_0/n}^{(n-1)/n} = a_{\lambda_{s+1}},$$

and therefore

$$c_{\lambda_{s+1} - [\lambda_0(n-1)/n]} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right).$$

Moreover, it is plain that  $\nu(\zeta_{\lambda_{s+1}}^n - \zeta) = \lambda_{s+2} > \lambda_{s+1}$ . Finally, note that all the exponents in  $X_{r+1}$  appearing in the development of  $\zeta_{\lambda_{s+1}}^n$  are of the type

$$\sum_{j=0}^s i_j \left[ L_j(\lambda_1, \dots, \lambda_j) - \beta_j \lambda_0 + \frac{\lambda_0}{n} \right] + i_{s+1} \left( \lambda_{s+1} - \lambda_0 + \frac{\lambda_0}{n} \right),$$

with  $i_0 + \dots + i_{s+1} = n$ . This implies that all these exponents are positive integers of the form  $L(\lambda_1, \dots, \lambda_{s+1}) - \gamma \lambda_0$ , with  $L$  a linear form with positive coefficients,  $\gamma \in \mathbf{N}$ .

In this way it is shown that

$$\sqrt[n]{\zeta} \in K' \left( \left( X_1^{\alpha_1/n} \right) \right) \dots \left( \left( X_r^{\alpha_r/n} \right) \right) \left[ \left[ X_{r+1}^{\lambda_0/n} \right] \right].$$

This finishes the proof of the proposition.



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