Hasse-Schmidt Derivations and Coefficient Fields in Positive Characteristics

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October 24, 2002

Abstract

We show how to express any Hasse-Schmidt derivation of an algebra in terms of a finite number of them under natural hypothesis. As an application, we obtain coefficient fields of the completion of a regular local ring of positive characteristic in terms of Hasse-Schmidt derivations.

KEYWORDS: Regular local ring, Coefficient field, Hasse–Schmidt derivation.
MSC:13H05, 13N15, 13N10, 13A35.

Introduction

Let $k \to A$ be a ring homomorphism. Hasse-Schmidt derivations of A over k are generalizations of usual derivations, but they do not carry an A-module structure. Nevertheless, Hasse-Schmidt derivations have a non abelian group structure lifting the addition of derivations.

In this paper we show how to express any Hasse-Schmidt derivation in terms of a finite number of them under very reasonable conditions. In proving our result, we find a natural way of producing "non-linear combinations" of Hasse-Schmidt derivations which, to some extent, could play the role of the A-module structure of derivations.

As an application, we express coefficient fields of the completion of a regular local ring of positive characteristic in terms of Hasse-Schmidt derivations, generalizing a similar result in characteristic zero.

Let us now comment on the content of this paper.

In section 1 we recall the notions of Hasse-Schmidt derivation and differential operator.

*Both authors are partially supported by DGI, BFM2001-3207.

Section 2 deals with the main result of this paper, namely that any Hasse-Schmidt derivation $\underline{\mathfrak{D}}$ can be expressed as "non-linear combination" of a finite number of them $\underline{D}^1, \ldots, \underline{D}^n$ whenever their degree 1 components $D_1^1, \ldots, D_1^n$ generate the module of usual derivations.

In section 3 we apply our main result to generalize a well known theorem of Nomura and to obtain coefficient fields of the completion of noetherian local regular rings in the positive characteristics case.

Our results seem related to some results in [8]. We hope to return to this relationship in a future work.

We thank Herwig Hauser for pointing out a gap in the statement of proposition (2.7) in an earlier version of this work.

1 Preliminaries and notations

All rings and algebras considered in this paper are assumed to be commutative with unit element.

(1.1) *Hasse-Schmidt derivations* (cf. [2] and [6], §27).

Let $k \xrightarrow{f} A \xrightarrow{g} B$ be ring homomorphisms. Let t be an indeterminate over B, and set $B_m = B[t]/(t^{m+1})$ for $m \geq 0$ and $B_\infty = B[[t]]$. We can view B_m as a k–algebra in a natural way (for $m \leq \infty$).

A *Hasse–Schmidt derivation* (over k) of length $m \geq 1$ (resp. of length ∞) from A to B, is a sequence $\underline{D} = (D_0, D_1, \ldots, D_m)$ (resp. $\underline{D} = (D_0, D_1, \ldots))$ of k–linear maps $D_i : A \longrightarrow B$, satisfying the conditions:

$$D_0 = g, \quad D_i(xy) = \sum_{r+s=i} D_r(x) D_s(y)$$

for all $x, y \in A$ and all $i > 0$. In particular, the first component D_1 is a k-derivation from A to B. Moreover, D_i vanishes on $f(k)$ for all $i > 0$.

Any Hasse-Schmidt derivation $\underline{D}$ is determined by a ring homomorphism

$$E : x \in A \mapsto \sum_{i=0}^{m} D_i(x) t^i \in B_m$$

with $E(x) \equiv g(x) \mod t$.

When $A = B$ and $g = 1_A$, we simply say that $\underline{D}$ is a Hasse-Schmidt derivation of A (over k). We write $\mathrm{HS}_k(A, B; m)$ for the set of all Hasse-Schmidt derivations (over k) of length m from A to B, $\mathrm{HS}_k(A, B) = \mathrm{HS}_k(A, B; \infty)$, $\mathrm{HS}_k(A; m) = \mathrm{HS}_k(A, A; m)$ and $\mathrm{HS}_k(A) = \mathrm{HS}_k(A, A; \infty)$.

We say that a k-derivation $\delta : A \to B$ is *integrable* [5] if there is a Hasse-Schmidt derivation $\underline{D} \in \mathrm{HS}_k(A, B)$ such that $D_1 = \delta$. The set of integrable k-derivations from A to B, denoted by $\mathrm{IDer}_k(A, B)$, is a submodule of the k-derivations B-module $\mathrm{Der}_k(A, B)$.

(1.2) *Differential operators* (cf. [1], §16, 16.8).

Let $f : k \to A$ be a ring homorphism.

For all $i \geq 0$, we inductively define the subsets $\mathcal{D}^{(i)}_{A/k} \subseteq \mathrm{End}_k(A)$ in the following way:

$$\mathcal{D}^{(0)}_{A/k} := A \subseteq \mathrm{End}_k(A), \quad \mathcal{D}^{(i+1)}_{A/k} := \{\varphi \in \mathrm{End}_k(A) \mid [\varphi, a] \in \mathcal{D}^{(i)}_{A/k}, \;\; \forall a \in A\}.$$

The elements of $\mathcal{D}_{A/k} := \bigcup_{i \geq 0} \mathcal{D}^{(i)}_{A/k}$ (resp. of $\mathcal{D}^{(i)}_{A/k}$) are called *linear differential operators* (resp. *linear differential operators of order* $\leq i$) of A/k. The family $\{\mathcal{D}^{(i)}_{A/k}\}_{i \geq 0}$ is an increasing sequence of (A, A)–bimodules of $\mathrm{End}_k(A)$ satisfying:

$$\mathcal{D}^{(1)}_{A/k} = A \oplus \mathrm{Der}_k(A), \quad \mathcal{D}^{(i)}_{A/k} \circ \mathcal{D}^{(j)}_{A/k} \subset \mathcal{D}^{(i+j)}_{A/k},$$

and $[P, Q] \in \mathcal{D}^{(i+j-1)}_{A/k}$ for all $P \in \mathcal{D}^{(i)}_{A/k}$, $Q \in \mathcal{D}^{(j)}_{A/k}$. Hence, $\mathcal{D}_{A/k}$ is a filtered subring of $\mathrm{End}_k(A)$. Moreover, linear differential operators of A/k are I–continuous for any I–adic topology. In particular, for any linear differential operator P of A/k, there is a unique extension $\widehat{P} \in \mathcal{D}_{\hat{A}/k}$ to the completion $\widehat{A}$ of A for any separated I–adic topology.

For each $\underline{D} \in \mathrm{HS}_k(A; m)$, one easily proves that $D_i \in \mathcal{D}^{(i)}_{A/k}$ and then there is a unique extension $\widehat{\underline{D}} \in \mathrm{HS}_k(\widehat{A}; m)$.

In a similar way, if $S \subset A$ is a multiplicatively closed subset, any Hasse-Schmidt derivation of A/k extends uniquely to a Hasse-Schmidt derivation of $S^{-1}A/k$.

(1.3) *Taylor expansions* (cf. [7]).

Let $n \geq 1$ be an integer. We write $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{T} = (T_1, \ldots, T_n)$, $\mathbf{X} + \mathbf{T} = (X_1 + T_1, \ldots, X_n + T_n)$ and, for $\alpha, \beta \in \mathbb{N}^n$, $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$ and $\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1} \cdots \binom{\beta_n}{\alpha_n}$.

We consider the usual partial ordering in $\mathbb{N}^n$: $\beta \geq \alpha$ means $\beta_1 \geq \alpha_1, \ldots, \beta_n \geq \alpha_n$. We write $\beta > \alpha$ if $\beta \geq \alpha$ and $\beta \neq \alpha$.

Let A be the formal power series ring $k[[\mathbf{X}]]$ (or the polynomial ring $A = k[\mathbf{X}]$). For any $f(\mathbf{X}) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \mathbf{X}^\alpha \in A$ we define $\Delta^{(\alpha)}(f(\mathbf{X}))$ by: $f(\mathbf{X} + \mathbf{T}) = \sum_{\alpha \in \mathbb{N}^n} \Delta^{(\alpha)}(f(\mathbf{X}))\mathbf{T}^\alpha$. It is well known that (cf. [1], §16, 16.11): $\Delta^{(\alpha)} \in \mathcal{D}^{(|\alpha|)}_{A/k}$, $\Delta^{(\alpha)}(f \cdot g) = \sum_{\beta + \sigma = \alpha} \Delta^{(\beta)}(f)\Delta^{(\sigma)}(g)$, $\alpha!\Delta^{(\alpha)} = (\frac{\partial}{\partial X_1})^{\alpha_1} \cdots (\frac{\partial}{\partial X_n})^{\alpha_n}$ and $\mathcal{D}^{(i)}_{A/k} = \bigoplus_{|\alpha| \leq i} A \cdot \Delta^{(\alpha)} = \bigoplus_{|\alpha| \leq i} \Delta^{(\alpha)} \cdot A$.

(1.4) If we denote $\Delta^{(0, \ldots, \overset{(j)}{i}, \ldots, 0)} = \Delta^j_i$, then $\underline{\Delta}^j = (1_A, \Delta^j_1, \Delta^j_2, \ldots) \in \mathrm{HS}_k(A)$.

Finally, let us recall the notion of quasi-coefficient field of a local ring.

(1.5) Definition. (cf. [4], 38.F) *Let $(R,\mathfrak{m})$ be a local ring, $K=R/\mathfrak{m}$ and k_0 a subfield of R. We say that k_0 is a* quasi–coefficient field *of R, if the extension K/k_0 is formally étale* (cf. [1], 17.1.1 and [4], 38.E).

In the case of characteristic 0, a field extension is formally étale if and only if it is separably algebraic *(cf. [4],* 38.*E).* On the other hand, any extension of perfect fields of characteristic $p>0$ is formally étale.

The following proposition is well known.

(1.6) Proposition. *Let $k_0 \to k \xrightarrow{f} A \xrightarrow{g} B$ be ring homomorphisms and let's suppose that the extension $k_0 \to k$ is formally étale. Then,* $\mathrm{HS}_{k_0}(A,B;m)=\mathrm{HS}_k(A,B;m)$ *for any integer $m\geq 1$ or $m=\infty$.*

2 Generating Hasse-Schmidt derivations

Throughout this section, let $k \xrightarrow{f} A$ be a ring homomorphism.

We consider the following partial ordering in $\mathbb{N}^n$: $\beta \succeq \alpha$ means that $\beta \geq \alpha$ and if $\alpha_i=0$ then $\beta_i=0$.

Let us denote by $\mathbb{N}_+$ de set of strictly positive integer numbers.

Let $N\geq 2$ be an integer and $\underline{\mathfrak{D}},\underline{D}^1,\ldots,\underline{D}^n \in \mathrm{HS}_k(A;N)$. For each $\mu\in\mathbb{N}^n$ we write $D_\mu = D^1_{\mu_1}\circ\cdots\circ D^n_{\mu_n}$. Let C_{ld} be elements in A, $1\leq d\leq n$, $1\leq l\leq N-1$, such that

$$\mathfrak{D}_i=\sum_{m=1}^{i}\left(\sum_{\substack{|\lambda|=i\\ |\mu|=m\\ \lambda\succeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_q d}\right)D_\mu \tag{1}$$

for all $i=1,\ldots,N-1$, where we write

$$\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_q d}=1 \quad \text{if} \quad \mu_d=\lambda_d=0. \tag{2}$$

Observe that the set $\mathbb{N}_+^0[=\mathbb{N}_+^{\emptyset}]$ has only one element and convention (2) follows by defining $|l|=0$ for $l\in\mathbb{N}_+^0$. Then we have $|l|\succeq r$ for any $r\geq 0$ and any $l\in\mathbb{N}_+^r$.

(2.7) Proposition. *Under the above hypothesis, the k-linear map*

$$\delta=\mathfrak{D}_N-\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\ |\mu|=m\\ \lambda\succeq\mu}}\prod_{d=1}^{n}\sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}}\prod_{q=1}^{\mu_d}C_{l_q d}\right)D_\mu$$

is a k-derivation of A.

PROOF. Let us take $a, b \in A$. Since

$$D_\mu(ab) = \sum_{\rho+\sigma=\mu} D_\rho(a) D_\sigma(b),$$

we obtain

$$\delta(ab) = \mathfrak{D}_N(a)b + a\mathfrak{D}_N(b) + \sum_{\substack{v+w=N\\ 1\le v,w\le N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) -$$

$$-\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\ |\mu|=m\\ \lambda\succeq\mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}\right)\cdot\left(\sum_{\rho+\sigma=\mu} D_\rho(a)D_\sigma(b)\right) =$$

$$= \mathfrak{D}_N(a)b + a\mathfrak{D}_N(b) + \sum_{\substack{v+w=N\\ 1\le v,w\le N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) -$$

$$-\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\ |\mu|=m\\ \lambda\succeq\mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}\right)\cdot (D_\mu(a)b + aD_\mu(b)) -$$

$$-\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\ |\mu|=m\\ \lambda\succeq\mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}\right)\cdot\left(\sum_{\substack{\rho+\sigma=\mu\\ \rho,\sigma>0}} D_\rho(a)D_\sigma(b)\right) =$$

$$= \delta(a)b + a\delta(b) + \sum_{\substack{v+w=N\\ 1\le v,w\le N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) -$$

$$-\sum_{m=2}^{N}\left(\sum_{\substack{|\lambda|=N\\ |\mu|=m\\ \lambda\succeq\mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d}\\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}\right)\cdot\left(\sum_{\substack{\rho+\sigma=\mu\\ \rho,\sigma>0}} D_\rho(a)D_\sigma(b)\right).$$

We need to prove that

$$\sum_{\substack{v+w=N\\ 1\le v,w\le N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) =$$

$$= \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succeq\mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_{+}^{\mu_d}\\|l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) \cdot \left(\sum_{\substack{\rho+\sigma=\mu\\\rho,\sigma>0}} D_{\rho}(a) D_{\sigma}(b) \right).$$

But

$$\sum_{\substack{v+w=N\\1\le v,w\le N-1}} \mathfrak{D}_v(a)\mathfrak{D}_w(b) =$$

$$= \sum_{\substack{v+w=N\\1\le v,w\le N-1}} \left[\sum_{r=1}^{v} \left(\sum_{\substack{|\tau|=v\\|\rho|=r\\\tau\succeq\rho}} \prod_{d=1}^{n} \sum_{\substack{l'\in\mathbb{N}_{+}^{\rho_d}\\|l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d} \right) D_{\rho}(a) \right] \cdot$$

$$\cdot \left[\sum_{s=1}^{w} \left(\sum_{\substack{|\omega|=w\\|\sigma|=s\\\omega\succeq\sigma}} \prod_{d=1}^{n} \sum_{\substack{l''\in\mathbb{N}_{+}^{\sigma_d}\\|l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d} \right) D_{\sigma}(b) \right] =$$

$$= \sum_{\substack{v+w=N\\1\le v,w\le N-1}} \sum_{m=2}^{N} \sum_{\substack{r+s=m\\1\le r\le v\\1\le s\le w}} \sum_{\substack{|\tau|=v,|\omega|=w\\|\rho|=r,|\sigma|=s\\\tau\succeq\rho,\omega\succeq\sigma}}$$

$$\left(\prod_{d=1}^{n} \sum_{\substack{l'\in\mathbb{N}_{+}^{\rho_d}\\|l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d} D_{\rho}(a) \right) \cdot \left(\prod_{d=1}^{n} \sum_{\substack{l''\in\mathbb{N}_{+}^{\sigma_d}\\|l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d} D_{\sigma}(b) \right) =$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N\\|\mu|=m\\\lambda\succeq\mu}} \sum_{\substack{\rho+\sigma=\mu\\\rho,\sigma>0}} \sum_{\substack{\tau+\omega=\lambda\\\tau\succeq\rho,\omega\succeq\sigma}}$$

$$\left(\prod_{d=1}^{n} \sum_{\substack{l'\in\mathbb{N}_{+}^{\rho_d}\\|l'|=\tau_d}} \prod_{q=1}^{\rho_d} C_{l'_q d} D_{\rho}(a) \right) \cdot \left(\prod_{d=1}^{n} \sum_{\substack{l''\in\mathbb{N}_{+}^{\sigma_d}\\|l''|=\omega_d}} \prod_{q=1}^{\sigma_d} C_{l''_q d} D_{\sigma}(b) \right) =$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \sum_{\substack{\tau+\omega=\lambda \\ \tau\succeq\rho,\omega\succeq\sigma}} \prod_{d=1}^{n} \sum_{\substack{l'\in\mathbb{N}_+^{\rho_d},|l'|=\tau_d \\ l''\in\mathbb{N}_+^{\sigma_d},|l''|=\omega_d}} \left(\prod_{q=1}^{\rho_d} C_{l'_q d} \cdot D_\rho(a) \right) \left(\prod_{q=1}^{\sigma_d} C_{l''_q d} \cdot D_\sigma(b) \right) =$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \prod_{d=1}^{n} \sum_{\substack{\tau_d+\omega_d=\lambda_d \\ \tau_d\succeq\rho_d,\omega_d\succeq\sigma_d}} \sum_{\substack{l'\in\mathbb{N}_+^{\rho_d},|l'|=\tau_d \\ l''\in\mathbb{N}_+^{\sigma_d},|l''|=\omega_d}} \left(\prod_{q=1}^{\rho_d} C_{l'_q d} \right) \left(\prod_{q=1}^{\sigma_d} C_{l''_q d} \right) D_\rho(a) D_\sigma(b) =$$

$$= \sum_{m=2}^{N} \sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d}=\mathbb{N}_+^{\rho_d}\times\mathbb{N}_+^{\sigma_d} \\ |l|=\lambda_d \\ [l=(l',l'')]}} \left(\prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\rho(a) D_\sigma(b) =$$

$$= \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) \cdot \left(\sum_{\substack{\rho+\sigma=\mu \\ \rho,\sigma>0}} D_\rho(a) D_\sigma(b) \right)$$

and the proposition is proved. Q.E.D.

(2.8) THEOREM. *Let $m \geq 1$ be an integer or $m = \infty$. Let $\underline{D}^1, \dots, \underline{D}^n \in \mathrm{HS}_k(A; m)$ be Hasse-Schmidt derivations of A/k such that their components of degree 1, $D_1^1, \dots, D_1^n$, form a system of generators of the A-module $\mathrm{Der}_k(A)$. Then, for any Hasse-Schmidt derivation $\underline{\mathfrak{D}} \in \mathrm{HS}_k(A; m)$ there exist $C_{ld} \in A$, $1 \leq d \leq n$, $1 \leq l < m+1$, such that the equation (1) holds for all $i \geq 1$. Moreover, if $\{D_1^1, \dots, D_1^n\}$ is a A-basis of $\mathrm{Der}_k(A)$, then the $\{C_{ld}\}$ are unique.*

PROOF. We proceed by induction on i.

For $i = 1$, $\mathfrak{D}_1$ is a derivation and so there exist $C_{11}, \dots, C_{1n} \in A$ such that

$$\mathfrak{D}_1 = C_{11} D_1^1 + \cdots + C_{1n} D_1^n.$$

Let N be an integer ≥ 2 and suppose we have elements $C_{ld} \in A$, $1 \leq l \leq N-1$, $1 \leq d \leq n$ such that relation (1) is true for $1 \leq i \leq N-1$.

By proposition (2.7), the k-linear map

$$\delta = \mathfrak{D}_N - \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l\in\mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu$$

is a k-derivation of A. Then, there exist $C_{Nd} \in A$, $1 \leq d \leq n$ s.t.

$$\delta = C_{N1} D_1^1 + \cdots + C_{Nn} D_1^n$$

and

$$\mathfrak{D}_N = \delta + \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu =$$

$$= \sum_{m=1}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d} \right) D_\mu$$

and equation (1) holds for $i = N$.

Obviously the C_{ld} are unique if $\{D_1^1, \ldots, D_1^n\}$ is a A-basis of $\mathrm{Der}_k(A)$. Q.E.D.

(2.9) Remark. The C_{ld} in theorem (2.8) depend on the order of the $\underline{D}^1, \ldots, \underline{D}^n$.

3 Applications: Coefficient Fields in positive characteristics

(3.10) Let $(R, \mathfrak{m}, k)$ be a noetherian regular local ring of dimension $n \geq 1$ containing a field, $X_1, \ldots, X_n \in \mathfrak{m}$ a regular system of parameters of R, $k_0 \subset R$ a quasi-coefficient field and $\widehat{R}$ the completion of R. We can identify $\widehat{R} = k[[X_1, \ldots, X_n]]$ by means of a canonical k_0-isomorphism. Let $\frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}$ be the usual basis of $\mathrm{Der}_k(\widehat{R})$ and $\underline{\Delta}^1, \ldots, \underline{\Delta}^n$ the Hasse–Schmidt derivations of $\widehat{R}$ over k defined in (1.4).

Let us recall the following result of M. Nomura ([3], Th. 2.3, [6], Th. 30.6)

(3.11) Theorem. *Under the hypothesis above, the following conditions are equivalent:*

(1) $\frac{\partial}{\partial X_i}(R) \subset R$ *for all* $i = 1, \ldots, n$.

(2) *There exist* $D_i \in \mathrm{Der}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$, *such that* $D_i(a_j) = \delta_{ij}$.

(3) *There exist* $D_i \in \mathrm{Der}_{k_0}(R)$ *and* $a_i \in R$, $i = 1, \ldots, n$ *such that* $\det(D_i(a_j)) \notin \mathfrak{m}$.

(4) $\mathrm{Der}_{k_0}(R)$ *is a free* R*–module of rank* n *(and* $\{D_1, \ldots, D_n\}$ *is a basis).*

(5) $\mathrm{rank}\,\mathrm{Der}_{k_0}(R) = n$.

The proof of the following corollaries are straightforward.

(3.12) COROLLARY. *Under hypothesis and equivalent conditions of theorem (3.11), for any basis* $D_1,\dots,D_n \in \mathrm{Der}_{k_0}(R)$*, their extensions* $\widehat{D}_1,\dots,\widehat{D}_n$ *to* $\widehat{R}$ *form a basis of* $\mathrm{Der}_k(\widehat{R})$*. Moreover, the restrictions* $\frac{\partial}{\partial X_i}|_R : R \to R$*,* $i=1,\dots,n$*, form a basis of* $\mathrm{Der}_{k_0}(R)$*.*

(3.13) COROLLARY. *Under hypothesis of corollary (3.12), let's suppose that* k_0 *is a field of characteristic* 0*. Then, the set* $\{a \in \widehat{R} \mid \widehat{D}_j(a) = 0 \quad \forall j = 1,\dots,n\}$ *is a coefficient field of* $\widehat{R}$ *(the only one containing* k_0*).*

The following theorem is an improvement of theorem (3.11) and is based on the results of section 2.

(3.14) THEOREM. *Under the hypothesis of (3.10), the following conditions are equivalent:*

(1) $\Delta_i^j(R) \subset R$*, for all* $j=1,\dots,n$*,* $i \geq 0$*.*

(2) There exist $\underline{D}^i \in \mathrm{HS}_{k_0}(R)$ *and* $a_i \in R$*,* $i=1,\dots,n$*, such that*

$$D_i^j(a_l) = \begin{cases} 0 & i \geq 2,\ \forall j \\ \delta_{jl} & i = 1,\ \forall j, l. \end{cases}$$

(3) There exist $D_i \in \mathrm{IDer}_{k_0}(R)$ *and* $a_i \in R$*,* $i=1,\dots,n$*, such that* $\det(D_j(a_l)) \notin \mathfrak{m}$*.*

(4) $\mathrm{Der}_{k_0}(R)$ *is a free* R*–module of rank* n *and* $\mathrm{IDer}_{k_0}(R) = \mathrm{Der}_{k_0}(R)$ *(and* $\{D_1,\dots,D_n\}$ *is a basis).*

(5) $\operatorname{rank} \mathrm{IDer}_{k_0}(R) = n$*.*

PROOF.
(1) $\Longrightarrow$ (2) $\Longrightarrow$ (3), (4) $\Longrightarrow$ (5) are straightforward.
(3) $\Longrightarrow$ (4) comes from theorem (3.11).

(5) $\Longrightarrow$ (1): Let $\underline{D}^1,\dots,\underline{D}^n \in \mathrm{HS}_{k_0}(R)$ such that $D_1^1,\dots,D_1^n$ are linear independent over R. Let us consider the extensions $\widehat{\underline{D}}^1,\dots,\widehat{\underline{D}}^n \in \mathrm{HS}_k(\widehat{R})$, whose degree 1 components $\widehat{D}_1^1,\dots,\widehat{D}_1^n$ are also linear independent over $\widehat{R}$.

Following the lines of the proof of theorem (2.8), we are going to prove the following result:
For any $j=1,\dots,n$, there exist $C_{ld}^j \in K = \mathrm{Q}(R)$, $1 \leq d \leq n$, $1 \leq l < +\infty$, such that

$$\Delta_i^j = \sum_{m=1}^{i} \left(\sum_{\substack{|\lambda|=i \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}^j \right) \widehat{D}_\mu \tag{3}$$

for all $i \geq 1$.

For[1] $i = 1$, the Δ_1^j are k-derivations of $\widehat{R}$ and then there exist $C_{1d}^j \in L = \mathrm{Q}(\widehat{R})$ such that

$$\Delta_1^j = \sum_{d=1}^{n} C_{1d}^j \widehat{D}_1^d.$$

Then, for any $m = 1, \ldots, n$ we have

$$\delta_{jm} = \Delta_1^j(X_m) = \sum_{d=1}^{n} C_{1d}^j \widehat{D}_1^d(X_m)$$

and the matrix $(\widehat{D}_1^j(X_m))$ with entries in R has a non-zero determinant. In particular $C_{1d}^j \in K$.

Let us suppose that for $N \geq 2$ there exist $C_{ld}^j \in K$, $1 \leq d, j \leq n$, $1 \leq l \leq N-1$, such that (3) holds for $i = 1, \ldots, N-1$. We can consider $\mathrm{HS}_k(\widehat{R}) \subset \mathrm{HS}_k(L)$. By proposition (2.7), for any $j = 1, \ldots, n$ the k-linear map

$$\delta^j = \Delta_N^j - \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} C_{l_q d}^j \right) \widehat{D}_\mu \tag{4}$$

is a k-derivation of L. Let $a \in R$ be a common denominator for the C_{ld}^j, $1 \leq d, j \leq n$, $1 \leq l \leq N-1$. Then,

$$a^N \delta^j = a^N \Delta_N^j - \sum_{m=2}^{N} \left(\sum_{\substack{|\lambda|=N \\ |\mu|=m \\ \lambda \succeq \mu}} \prod_{d=1}^{n} \sum_{\substack{l \in \mathbb{N}_+^{\mu_d} \\ |l|=\lambda_d}} \prod_{q=1}^{\mu_d} (a^{l_q} C_{l_q d}^j) \right) \widehat{D}_\mu$$

maps $\widehat{R}$ into $\widehat{R}$ and $a^N \delta^j \in \mathrm{Der}_k(\widehat{R})$. There exist $\overline{C}_{Nd}^j \in L$, $1 \leq d, j \leq n$, such that

$$a^N \delta^j = \sum_{d=1}^{n} \overline{C}_{Nd}^j \widehat{D}_1^d.$$

Since the matrix $(\widehat{D}_1^j(X_m))$ with entries in R has a non-zero determinant and $(a^N \delta^j)(X_m) \in R$ (notice that $\Delta_N^j(X_m) = 0$), we deduce that $\overline{C}_{Nd}^j \in K$. By setting $C_{Nd}^j = a^{-N} \overline{C}_{Nd}^j \in K$ we obtain the expression (3) for $i = N$.

[1]This is the same argument used in the proof of theorem (3.11).

From (3) we deduce that

$$\Delta_i^j(R) \subset K \cap \widehat{R} = R,$$

for all $j = 1, \ldots, n$, $i \geq 0$, and (1) is proved. Q.E.D.

(3.15) REMARK. As noticed in [4], page 289 for theorem (3.11), theorem (3.14) also holds for $\mathrm{HS}_k(\widehat{R}) \cap \mathrm{HS}(R)$ instead of $\mathrm{HS}_{k_0}(R)$, $\mathrm{Der}_k(\widehat{R}) \cap \mathrm{Der}(R)$ instead of $\mathrm{Der}_{k_0}(R)$ and $\{\delta \in \mathrm{Der}(R) \mid \exists \underline{D} \in \mathrm{HS}_k(\widehat{R}) \cap \mathrm{HS}(R) \text{ s.t. } D_1 = \delta\}$ instead of $\mathrm{IDer}_{k_0}(R)$, and the mention to a quasi-coefficient field can be avoided.

(3.16) REMARK. We do not know any example of a noetherian regular local ring $(R, \mathfrak{m}, k)$ containing a quasi-coefficient field k_0 (of positive characteristic) such that $\mathrm{IDer}_{k_0}(R) \neq \mathrm{Der}_{k_0}(R)$.

The following theorem generalizes corollary (3.13) to the case of characteristic $p \geq 0$.

(3.17) THEOREM. *Under the hypothesis of (3.10), let* $\underline{D}^1, \ldots, \underline{D}^n \in \mathrm{HS}_{k_0}(R)$ *such that their degree 1 components* $\{D_1^1, \ldots, D_1^n\}$ *form a basis of* $\mathrm{Der}_{k_0}(R)$. *Let* $\widehat{\underline{D}}^1, \ldots, \widehat{\underline{D}}^n$ *be the extensions of* $\underline{D}^1, \ldots, \underline{D}^n$ *to* $\widehat{R}$. *Then, the set* $\{a \in \widehat{R} \mid \widehat{D}_i^j(a) = 0 \quad \forall j = 1, \ldots, n,\ i \geq 1\}$ *is a coefficient field of* $\widehat{R}$ *(the only one containing* k_0*).*

PROOF. Since $\widehat{R} = k[[X_1, \ldots, X_n]]$, we have $k = \{a \in \widehat{R} \mid \Delta_i^j(a) = 0 \quad j = 1, \ldots, n;\ i \geq 1\}$. By corollary (3.12) we deduce that $\{\widehat{D}_1^1, \ldots, \widehat{D}_1^n\}$ is a $\widehat{R}$–basis of $\mathrm{Der}_k(\widehat{R})$, and from theorem (2.8) we can express the Δ_i^j in terms of $\widehat{D}_i^j$. In particular

$$\{a \in \widehat{R} \mid \widehat{D}_i^j(a) = 0 \quad \forall j = 1, \ldots, n,\ i \geq 1\} \subset k.$$

The opposite inclusion comes from proposition (1.6). Q.E.D.

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