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F. J. Herrera Govantes, M. A. Olalla Acosta, M. Spivakovsky, B. Teissier

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Departamento de Álgebra. Universidad de Sevilla

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F. J. Herrera Govantes
Departamento de Álgebra
Facultad de Matemáticas
Avda. Reina Mercedes, s/n
Universidad de Sevilla
41012 Sevilla, Spain

M. A. Olalla Acosta*
Departamento de Álgebra
Facultad de Matemáticas
Avda. Reina Mercedes, s/n
Universidad de Sevilla
41012 Sevilla, Spain

M. Spivakovsky
Laboratoire de Mathématiques E. Picard
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse cedex 4, France.

B. Teissier
Equipe " Géométrie et Dynamique",
Institut mathématique de Jussieu
175 Rue du Chevaleret
F-75013 Paris, France.

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1 Introduction

All the rings in this paper will be commutative with 1.

Let (R, m, k) be a local noetherian domain with field of fractions K and $\nu : K^* \to \Gamma$ a valuation of K, centered at R. Let $\hat{R}$ denote the m-adic completion of R (which, of course, need not in general be a domain). In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\hat{R}$ and ν by a suitable extension $\hat{\nu}$ to $\frac{\hat{R}}{P}$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$ (below, we will mention two specific applications we have in mind). It is well known and not hard to prove that such extensions $\hat{\nu}$ exist for some minimal prime ideals P of $\hat{R}$. In general, such a $\hat{\nu}$ is far from being unique. The purpose of our work is to give, assuming that R is a G-ring (see the definition below), a systematic description of all such extensions $\hat{\nu}$ and to identify certain classes of extensions which are of particular interest for applications. In this paper we will construct the implicit ideals of ν. These ideals will be very useful to describe such extensions $\hat{\nu}$, in a future paper.

When studying the extensions of ν to the completion of R, one is led to the study of its extensions to the henselization $\tilde{R}$ of R as a natural first step. We therefore start out by letting $\sigma : R \to R^\dagger$ denote one of the two operations of henselization or completion:

$$R^\dagger = \hat{R} \text{ or} \quad (1)$$

$$R^\dagger = \tilde{R}, \quad (2)$$

with R a G-ring.

Let r denote the (real) rank of ν. Let $(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \cdots \subsetneqq \Delta_0 = \Gamma$ be the isolated subgroups of Γ and $P_0 = (0) \subset P_1 \subset \cdots \subset P_r = m$ the prime valuation ideals of R, which

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need not, in general, be distinct. In this paper, we will assume that R is a G-ring. Under this assumption, we will canonically associate to ν a chain $H_1 \subset H_3 \subset \cdots \subset H_{2r+1} = mR^\dagger$ of ideals of $R^\dagger$, numbered by odd integers from 1 to $2r+1$, such that $H_{2\ell+1} \cap R = P_\ell$ for $0 \le \ell \le r$. We will show that all the ideals $H_{2\ell+1}$ are prime. We will define $H_{2\ell}$ to be the unique minimal prime ideal of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$ (that such a minimal prime is unique follows from the regularity of the homomorphism σ).

We will thus obtain, in the cases (1) and (2), a chain of $2r+1$ prime ideals

$$H_0 \subset H_1 \subset \cdots \subset H_{2r} = H_{2r+1} = mR^\dagger,$$

satisfying $H_{2\ell} \cap R = H_{2\ell+1} \cap R = P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell R^\dagger$ for $0 \le \ell \le r$. Moreover, if $R^\dagger = \tilde{R}$, then $H_{2\ell} = H_{2\ell+1}$. We call H_ℓ the **ℓ-th implicit prime ideal** of $R^\dagger$, associated to R and ν. The ideals H_ℓ behave well under local blowing ups along ν (that is, birational local homomorphisms $R \to R'$, essentially of finite type, such that ν is centered at R'). This means that given any local blowing up $R \to R'$ along ν the l-th implicit prime ideal H'_ℓ of ${R'}^\dagger$ has the property that $H'_\ell \cap \hat{R} = H_\ell$.

For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$.

Let $(0) \subsetneqq \mathbf{m}_1 \subsetneqq \cdots \subsetneqq \mathbf{m}_{r-1} \subsetneqq \mathbf{m}_r = \mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν. By definitions, our valuation ν is a composition of r rank one valuations $\nu = \nu_1 \circ \nu_2 \cdots \circ \nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$.

If $R^\dagger = \tilde{R}$, we will prove that there is a unique extension $\tilde{\nu}$ of ν to $\frac{R^\dagger}{H_0}$. If $R^\dagger = \hat{R}$, the situation is more complicated. First, we need to discuss the behaviour of our constructions under local blowings up with respect to ν (that is, birational local homomorphisms $R \to R'$, essentially of finite type, such that ν is centered at R').

1.1 Local blowings up and trees.

All the local blowings $R \to R'$ considered in this paper will be with respect to ν (that is, ν will always be centered in the local ring R'). Such local blowings up form a direct system $\{R'\}$. In this paper, we will consider many direct systems of rings and of ideals indexed by $\{R'\}$. Direct limits will always be taken with respect to the direct system R', unless otherwise specified.

Definition 1.1 *A **tree** of R'-algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R'-algebra. We have the obvious notion of homomorphism of trees.*

For example, both $\{\hat{R}'\}$ are trees of and $\{\tilde{R}'\}$ are trees of R'-algebras.

Definition 1.2 *Let $\{S'\}$ be a **tree** of R'-algebras. Let I' be an ideal of S'. We say that $\{I'\}$ is a tree of ideals if for any arrow $S' \to S''$ in our direct system, we have $I'' \cap S' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.*

Examples. For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$. For any non-negative element $\beta \in \Gamma$, the valuation ideals $P'_\beta \subset R'$ of value β form a tree of ideals of R'. Similarly, the i-th prime valuation ideals $P'_i \subset R'$ form a tree. If $rk\ \nu = r$, the prime valuation ideals P'_i give rise to a chain

$$P'_0 = (0) \subset P'_1 \subset \cdots \subset P'_r = m \qquad (3)$$

of trees of prime ideals of R'.

We discuss this last example in a little more detail and generality in order to emphasize our point of view, crucial throughout this paper: the data a composite valuation is equivalent to the data of its components. Namely, suppose we are given a chain of trees of ideals as in (3), where we relax our assumptions of the P'_i as follows. We no longer assume that the chain (3) is maximal, nor that $P'_i \subsetneqq P'_{i+1}$, even for R' sufficiently large; in particular, for the purposes of this example we momentarily drop the assumption that $rk\ \nu = r$. We will still assume, however, that $P'_0 = (0)$ and that $P'_r = m'$.

Taking the limit in (3), we obtain a chain

$$(0) = \mathbf{m}_0 \subsetneqq \mathbf{m}_1 \subsetneqq \dots \subsetneqq \mathbf{m}_r = \mathbf{m}_\nu \tag{4}$$

of prime ideals of the valuation ring R_ν.

Then specifying the valuation ν is equivalent to specifying valuations $\nu_0, \nu_1, \dots, \nu_r$, where ν_0 is the trivial valuation of K and, for $1 \le l \le r$, ν_l is a valuation of the residue field $k_{\nu_{l-1}} = \kappa(\mathbf{m}_{l-1})$, centered at the local ring $\lim\limits_{\to} \frac{R'_{P'_l}}{P'_{l-1}} = \frac{(R_\nu)_{\mathbf{m}_l}}{\mathbf{m}_{l-1}}$.

The relationship between ν and the ν_l is that ν is the composition $\nu = \nu_1 \circ \nu_2 \circ \dots \circ \nu_r$. If we assume, in addition, that the chain (3) (equivalently, (4)) is a maximal chain of distinct prime ideals then $rk\ \nu = r$ and $rk\ \nu_l = 1$ for each l.

Coming back to the implicit prime ideals, we will see that the implicit prime ideals H'_i form a tree of ideals of $R^\dagger$.

We will show that if ν extends to a valuation of $\hat{\nu}$ centered at $\frac{\hat{R}}{P}$ with $P \cap R = (0)$ then the prime P must contain the minimal prime H_0 of $\hat{R}$. We will then show that specifying an extension $\hat{\nu}$ of ν as above is equivalent to specifying a chain of prime valuation ideals $\tilde{H}'_0 \subset \tilde{H}'_1 \subset \dots \subset \tilde{H}'_{2r} = m'\hat{R}'$ such that $H'_\ell \subset \tilde{H}'_\ell$ for all $\ell \in \{0, \dots, 2r\}$, and valuations $\hat{\nu}_1, \hat{\nu}_2, \dots, \hat{\nu}_{2r}$, where $\hat{\nu}_j$ is a valuation of the field $\kappa(\hat{\mathbf{m}}_{j-1})$, arbitrary when j is odd and satisfying certain conditions, coming from the valuation $\nu_{\frac{j}{2}}$, when j is even. Here $\hat{\mathbf{m}}_{j-1}$ denotes the $(j-1)$-st prime ideal of the valuation ring $R_{\hat{\nu}}$; its intersection with $\hat{R}'$ is $\tilde{H}'_{j-1}$.

The prime ideals H_j are defined as follows. Recall that given a valued ring (R, ν) with value group Γ, that is a subring $R \subseteq R_\nu$ of the valuation ring R_ν of a valuation of the field of fractions K of R with group Γ, one defines for each $\gamma \in \Gamma$ the valuation ideals of R associated to γ as follows:

$$\begin{aligned} \mathcal{P}_\gamma(R) = &\quad \{x \in R / \nu(x) \ge \gamma\} \\ \mathcal{P}^+_\gamma(R) = &\quad \{x \in R / \nu(x) > \gamma\} \end{aligned}$$

and the associated graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\gamma \in \Gamma} \frac{\mathcal{P}_\gamma(R)}{\mathcal{P}^+_\gamma(R)} = \bigoplus_{\gamma \in \Gamma_+ \cup \{0\}} \frac{\mathcal{P}_\gamma(R)}{\mathcal{P}^+_\gamma(R)}.$$

The second equality comes from the fact that if $\gamma \in \Gamma_-$, we have $\mathcal{P}^+_\gamma(R) = \mathcal{P}_\gamma(R) = R$. If $R \to R'$ is a birational extension of local rings such that $R \subset R' \subset R_\nu$ and $m_\nu \cap R' = m'$ that is, a local blowing up along ν, we may write $\mathcal{P}'$ for $\mathcal{P}(R')$.

We now define

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta R'^\dagger \right) \bigcap R^\dagger \right), \tag{5}$$

where R' ranges over all the local blowings up of R along ν.

The questions answered in this paper originally arose from our work on the Local Uniformization Theorem, where passage to completion is required in both the approaches of [8] and

[9]. In [9], one really needs to pass to completion for valuations of arbitrary rank. One of the main applications of the theory of implicit prime ideals will be the following result, announced in [9].

Theorem 1.1 *With the above notation, assume that the Local Uniformization Theorem holds for all the local rings of dimension at most* dim $R-1$*, which are birational and essentially of finite type over* $\frac{\hat{R}}{P}$*, where* P *is some prime ideal of* $\hat{R}$ *such that* $P\cap R=(0)$*. Then there exists a tree of prime ideals* H' *of* $\hat{R}'$ *with* $H'\cap R'=(0)$ *and a valuation* $\hat{\nu}$ *with value group* Γ*, centered at* $\lim\limits_{\to}\frac{\hat{R}'}{H'}$ *and having the following property. For any local blowing up* R' *there exists a local blowing up* $R'\to R''$ *such that the graded algebra* $gr_{\hat{\nu}}\frac{\hat{R}''}{H''}$ *is scalewise birational to* $gr_\nu R''$ *(the definition of scalewise birational will be recalled below).*

As observed in [9], there is no danger of circular reasoning when applying this result to the Local Uniformization Theorem, since one can proceed by induction on dim R. In the above Theorem, local uniformization is assumed up to dimension dim $R-1$ and it is used for local uniformization in dimension dim R.

The approach to the Local Uniformization Theorem taken in [8] is to reduce the problem to the case of rank 1 valuations. The theory of implicit prime ideals is much simpler for valuations of rank 1 and takes only a few pages in [8].

Another recent application of these results on extending valuations to the formal completion will be the work [4] on the Pierce-Birkhoff conjecture. This conjecture was reduced by J. Madden to studying the separating ideal $<\alpha,\beta>$ of two points α and β in the real spectrum of A (of course A must be of characteristic 0 in order for Sper A to be non-empty). It is easy to reduce the problem to the case when A is local. However, for various technical reasons one is induced to pass to the formal completion $\hat{A}$ of A. At this point, we need to know that there exist points $\hat{\alpha}$ and $\hat{\beta}$ in the real spectrum of $\hat{A}$ such that $<\hat{\alpha},\hat{\beta}>\cap A=<\alpha,\beta>$. This fact is deduced readily from 1.1.

The paper is organized as follows. In §2 we define the odd-numbered implicit ideals $H_{2\ell+1}$ and prove that $H_{2\ell+1}\cap R=P_\ell$. We observe that by the very definition, the ideals $H_{2\ell+1}$ behave well under local blowings up along ν. Proving that $H_{2\ell+1}$ is indeed prime is postponed until §4, since the proof of this for $\hat{R}$ uses the uniqueness of the extension $\tilde{\nu}$ of ν to $\tilde{R}$ modulo its first implicit prime ideal. We also give two examples to motivate the definition 5 – one to explain the need to localize at $H_{2\ell+1}$, the other the need of taking the limit with respect to R'.

In §3 we prove that $H_{2\ell+1}$ is prime for the henselization. We study the extension of ν to $\tilde{R}$ modulo its first implicit prime ideal and prove that such an extension is unique.

In §4 we prove that $H_{2\ell+1}$ are indeed prime for the completion and that $H_{2\ell}\cap R=H_{2\ell+1}\cap R=P_\ell$.

It follows from the noetherianity of $R^\dagger$, that there exists a specific R' for which the limit is achieved. In §3 and §4 we explain in more detail what that R' is; it turns out that the R' which has this property for the henselization, automatically has it for the completion.

In the future we will describe the set of extensions $\hat{\nu}$ to $\frac{\hat{R}}{P}$, where P is a prime ideal such that $P\cap R=(0)$. We will describe one class of such extensions, which we call "natural", for which $\tilde{H}_j=H_j$ for all j and the valuation $\hat{\nu}_{2\ell+1}$ is completely determined by ν_ℓ for each ℓ. Up until this point in the paper the Local Uniformization Theorem has not yet appeared. Finally, we will use the Local Uniformization Theorem to prove Theorem 1.1. We would like to acknowledge the paper [1] by Bill Heinzer and Judith Sally which inspired one of the authors to continue thinking about this subject.

2 Definition and first properties of the implicit ideals.

Let the notation be as above. We define our main object of study, the j-th implicit prime ideal H_j as follows. First we put

$$H_{2r} = H_{2r+1} = mR^\dagger.$$

Now let ℓ be a strictly positive integer, $\ell \leq r$. Put

$$H_{2\ell-1} = \bigcap_{\beta \in \Delta_{\ell-1}} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta R'^\dagger \right) \bigcap R^\dagger \right), \tag{6}$$

where R' ranges over all the local blowings up of R along ν. The ideal P'_β is, by definition, the valuation ideal of R' of value β. This ideal contains $P_\beta R'$ but may, in general, be larger.

The following two examples illustrate the need for taking the limit over the local blowings up R'.

Example. Let us consider the local domain $S = \frac{k[x,y]_{(x,y)}}{(y^2-x^2-x^3)}$. There are two valuations centered in (x, y). Let $a_i \in k,\ i \geq 2$ be such that

$$\left(y + x + \sum_{i\geq 2} a_i x^i \right) \left(y - x - \sum_{i \geq 2} a_i x^i \right) = y^2 - x^2 - x^3.$$

We shall denote ν_+ to the rank one discrete valuation defined by

$$\nu_+(x) = \nu_+(y) = 1,$$

$$\nu_+(y + x) = 2,$$

$$\nu_+ \left(y + x + \sum_{i \geq 2}^{m-1} a_i x^i \right) = m.$$

Now let $R = \frac{k[x,y,z]_{(x,y,z)}}{(y^2-x^2-x^3)}$. Let $\Gamma = \mathbb{Z}^2$ with the lexicographical ordering. Let ν be the composite valuation of the (z)-adic one with ν_+ centered in $R/(z)$. In this example we are suppossing that

$$H_{2\ell+1} = \cap_{\beta \in \Delta_\ell} P_\beta \hat{R}.$$

As the ideal $P_{(n,0)} = (z^n)$ then

$$H_1 = \cap_{(n,m) \in \mathbb{Z}^2} P_{(n,m)} \hat{R} = (0).$$

Let $f = y+x+\sum_{i\geq 2} a_i x^i$ and $g = y-x-\sum_{i \geq 2} a_i x^i$ be elements of $\hat{R}$. Clearly $f, g \notin H_1 = (0)$, but $f \cdot g = (0)$ and the ideal H_1 is not prime. If we consider localization in order to define the implicit ideals,

$$H_{2\ell+1} = \left(\cap_{\beta \in \Delta_\ell} P_\beta \hat{R}_{H_{2\ell+3}} \right) \cap \hat{R},$$

then $H_1 = (f)$ is prime.

Example. Let $R = \frac{k[x,y,z]_{(x,y,z)}}{(z^2-y^2(1+x))}$. Let $\Gamma = \mathbb{Z}^2$ with the lexicographical ordering. Let t be an independent variable and let ν be the valuation, centered in R, induced by the t-adic valuation of $k[[t^\Gamma]]$ under the injective homomorphism $\iota : R \hookrightarrow k[[t^\Gamma]]$, defined by $\iota(x) = t^{(0,1)}$, $\iota(y) = t^{(1,0)}$

and $\iota(z) = t^{(1,0)}\sqrt{1+t^{(0,1)}}$. Then we have $\bigcap\limits_{\beta\in\Phi} P_\beta\hat{R} = (0)$ and $\bigcap\limits_{\beta\in\Phi\cap\Delta_1} P_\beta\hat{R} = (y,z)\hat{R} = P_1\hat{R}$, where the ideal (0) is not prime in $\hat{R}$. Now, let $R' = R\left[\frac{z}{y}\right]_{M'}$, where $M' = \left(x, y, \frac{z}{y} - 1\right)$ is the center of ν in $R\left[\frac{z}{y}\right]$. We have $z - y\sqrt{1+x} \in P'_\beta\hat{R}' \setminus P_{(2,0)\hat{R}}$; in particular, $z - y\sqrt{1+x} \in \bigcap\limits_{\beta\in\Phi} P'_\beta\hat{R}' \setminus P_{(2,0)\hat{R}}$. Thus this example also shows that the ideals $P_\beta\hat{R}$ and $\bigcap\limits_{\beta\in\Phi} P_\beta\hat{R}$ do not behave well under blowing up.

Note that both of these examples occur not only for the completion $\hat{R}$ but also for the henselization $\tilde{R}$.

Proposition 2.1 *We have $H_{2\ell-1} \cap R = P_{\ell-1}$.*

Proof: Recall that $\mathcal{P}_{\ell-1} = \{x \in R / \ \nu(x) \notin \Delta_{\ell-1}\}$. If $x \in \mathcal{P}_{\ell-1}$ then, because $\Delta_{\ell-1}$ is an isolated subgroup, we have $x \in \mathcal{P}_\beta$ for all $\beta \in \Delta_{\ell-1}$. The same inclusion holds for the same reason in all extensions $R' \subset R_\nu$ of R, and this implies the inclusion $\mathcal{P}_{\ell-1} \subseteq H_{2\ell-1} \cap R$. Let now x be in $H_{2\ell-1} \cap R$ and assume $x \notin \mathcal{P}_{\ell-1}$. Then there is a $\beta \in \Delta_{\ell-1}$ such that $x \notin \mathcal{P}_\beta$. By the faithful flatness of $\hat{R}$ over R this implies $x \notin \mathcal{P}_\beta\hat{R}$ since $\mathcal{P}_\beta\hat{R} \cap R = \mathcal{P}_\beta$, and the same holds in all extensions $R' \subset R_\nu$ of R so that x cannot be in $H_{2\ell-1} \cap R$. This contradiction shows the desired equality.$\square$

Proposition 2.2 *The ideals $H'_{2\ell-1}$ behave well under local blowings up along ν. In other words, let $R \to R'$ be a local blowing up along ν and let $H'_{2\ell-1}$ denote the $(2l-1)$-st implicit prime ideal of $\hat{R}'$. Then $H_{2\ell-1} = H'_{2\ell-1} \cap R^\dagger$.*

Proof: Immediate from the definitions. $\square$

To study the ideals $H_{2\ell+1}$, we need to understand more explicitly the nature of the limit appearing in (6). To study the relationship between the ideals $P_\beta R^\dagger$ and $P'_\beta R'^\dagger \bigcap R^\dagger$, it is useful to factor the natural map $R^\dagger \to R'^\dagger$ as $R^\dagger \to R^\dagger \otimes_R R' \xrightarrow{\phi} R'^\dagger$. In general, the ring $R^\dagger \otimes_R R'$ is not local (see the above examples), but it has one distinguished maximal ideal M', namely, the ideal generated by $mR^\dagger \otimes 1$ and $1 \otimes m'$, where m' denotes the maximal ideal of R'. The map ϕ factors through the local ring $\left(R^\dagger \otimes_R R'\right)_{M'}$ and the resulting map $\left(R^\dagger \otimes_R R'\right)_{M'} \to R'^\dagger$ is either the formal completion or the henselization; in either case, it is faithfully flat. Thus $P'_\beta R'^\dagger \cap \left(R^\dagger \otimes_R R'\right)_{M'} = P'_\beta\left(R^\dagger \otimes_R R'\right)_{M'}$. This shows that we may replace $R'^\dagger$ by $\left(R^\dagger \otimes_R R'\right)_{M'}$ in (6) without affecting the result, that is,

$$H_{2\ell-1} = \bigcap_{\beta\in\Delta_{\ell-1}} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta \left(R^\dagger \otimes_R R'\right)_{M'}\right) \bigcap R^\dagger\right). \tag{7}$$

From now on, we will use (7) as our working definition of the implicit prime ideals. One advantage of the expression (7) is that it makes sense in a situation more general than the completion and the henselization. Namely, to study the case of the henselization $\tilde{R}$, we will need to consider étale extensions $R^\dagger$ of R, which are contained in $\tilde{R}$ (particularly, those which are essentially of finite type). The definition (7) of the implicit prime ideals makes sense also in that case.

3 Extending a valuation centered in a local G-domain to its henselization.

Let $\tilde{R}$ denote the henselization of R, as above. The completion homomorphism $R \to \hat{R}$ factors through the henselization: $R \to \tilde{R} \to \hat{R}$. In this section, we will show that H_1 a minimal prime of $\tilde{R}$, that ν extends uniquely to a valuation $\tilde{\nu}$ of rank r centered at $\frac{\tilde{R}}{H_1}$, and that H_1 is the unique prime ideal P of $\tilde{R}$ such that ν extends to a valuation of $\frac{\tilde{R}}{P}$. Furthermore, we will prove that $H_{2\ell+1}$ is a minimal prime of $P_\ell\tilde{R}$ for all l and that these are precisely the prime $\tilde{\nu}$-ideals of $\tilde{R}$.

Studying the implicit prime ideals of $\tilde{R}$ and the extension of ν to $\tilde{R}$ is a logical intermediate step before attacking the formal completion, for the following reason. As we will show in the next section, if R is already henselian in (6) then $P'_\beta\hat{R}'_{H'_{2\ell+1}}\cap\hat{R}=P_\beta\hat{R}$ for all β and R' and thus we have that $H_{2\ell-1}=\bigcap\limits_{\beta\in\Delta_{\ell-1}}\left(\mathcal{P}_\beta\hat{R}\right)$.

We state the main result of this section. In the case when $R^\dagger$ is an étale extension of R, contained in $\tilde{R}$, we use (7) as our definition of the implicit prime ideals.

Theorem 3.1 *Let $R^\dagger$ be a local étale extension of R, contained in $\tilde{R}$. Then:*

(1) The ideal $H_{2\ell+1}$ is prime for $0\le l\le r$; it is a minimal prime of P_ℓ. In particular, H_1 is a minimal prime of $R^\dagger$.

(2) The ideal H_1 is the unique prime P of $R^\dagger$ such that there exists an extension $\nu^\dagger$ of ν to $\frac{R^\dagger}{H_1}$; the extension $\nu^\dagger$ is unique. The graded algebra $gr_{\nu^\dagger}\frac{R^\dagger}{H_1}$ is scalewise birational to $gr_\nu R$; in particular, $rk\ \nu^\dagger=r$.

(3) The ideals $H_{2\ell+1}$ are precisely the prime $\nu^\dagger$-ideals of $R^\dagger$.

Proof: By assumption, the ring $R^\dagger$ is a direct limit of local, strict étale extension of R which are essentially of finite type. All the assertions (1)-(3) behave well under taking direct limits, so it is sufficient to prove the Theorem in the case when $R^\dagger$ is essentially of finite type over R. From now on, we will restrict attention to this case.

The next step is to characterize explicitly those local blowings up $R\to R'$ for which the limit in (7) is attained.

Since $R^\dagger$ is algebraic, essentially of finite type over R, the ring $\kappa(P_\ell)\otimes_R R^\dagger$ is finite over $\kappa(P_\ell)$; this ring is reduced but it may contain zero divisors. In fact, it is known that the minimal primes of $\frac{\tilde{R}}{P_\ell\tilde{R}}$ (and of $\frac{\hat{R}}{P_\ell\hat{R}}$) correspond one-to-one to maximal ideals of the normalization of $\frac{R}{P_\ell}$ (here we use the fact that R is a G-ring). Let Λ denote the set of minimal primes of $\frac{\tilde{R}}{P_\ell\tilde{R}}$. Since we have strict étale maps $\frac{R}{P_\ell}\to\frac{R^\dagger}{P_\ell R^\dagger}\to\frac{\tilde{R}}{P_\ell\tilde{R}}$, there exists a partition $\Lambda=\coprod\limits_{i=1}^{s}\Lambda_i$ such that the sets Λ_i are in one-to-one correspondence with the minimal primes of $\frac{R^\dagger}{P_\ell R^\dagger}$. Now, let N denote the center of the valuation induced by ν on $\frac{R}{P_\ell}$ in the normalization S of $\frac{R}{P_\ell}$. Since N lies over $\frac{m}{P_\ell}$, it is a maximal ideal of S. The ring $S_N\otimes_R R^\dagger$ is contained in the henselization of the normal local G-ring S_N and is therefore an integral domain. Thus the zero ideal of $S_N\otimes_R R^\dagger$ contracts to a certain minimal prime of $\frac{R^\dagger}{P_\ell R^\dagger}$; let us denote this minimal prime by Q_0. Since the ring $\kappa(P_\ell)\otimes_R R^\dagger$ is reduced and finite over $\kappa(P_\ell)$, it is a direct product of fields, each of which is finite over $\kappa(P_\ell)$. These fields are precisely the fields of fractions of the rings of the form $\frac{R^\dagger}{P_\ell R^\dagger}/Q$, where Q ranges over the minimal primes of $\frac{R^\dagger}{P_\ell R^\dagger}$. Let $L=L_0$ be the field of fractions of $\frac{R^\dagger}{P_\ell R^\dagger}/Q_0$; L is a finite extension of $\kappa(P_\ell)$.

Now let $\pi:R\to R'$ be a local blowing up of R along ν. Let $\kappa(P_\ell)\to\kappa(P'_\ell)$ be the field extension induced by π. Let d be the greatest integer such that for some local blowing up π, L contains an extension of $\kappa(P_\ell)$ of degree d, isomorphic to some extension of $\kappa(P_\ell)$, contained in $\kappa(P'_\ell)$.

Lemma 3.1 *Fix an integer $l \in \{1, \dots, r\}$. There exists a local blowing up $R \to R'$ along ν having the following property. Let P'_ℓ denote the ℓ-th prime ν-ideal of R'. Then the ring $\frac{R'}{P'_\ell}$ is analytically irreducible; in particular, $\frac{R'}{P'_\ell} \otimes R^\dagger$ is an integral domain.*

Proof of Lemma 3.1: Since R is a local G-ring, every homomorphic image of R is Nagata [5]. Let $\pi : \frac{R}{P_\ell} \to S$ be the normalization of $\frac{R}{P_\ell}$ and let $\rho : R \to R''$ be a birational homomorphism such that Spec S is the (scheme-theoretic) inverse image of $\frac{R}{P_\ell}$ in Spec R''. Explicitly, such a map ρ can be described as follows. Write S as $S = \frac{R}{P_\ell}\left[\frac{\bar{a}_1}{\bar{b}_1}, \dots, \frac{\bar{a}_n}{\bar{b}_n}\right]$, where $\bar{a}_i, \bar{b}_i$ are elements of $\frac{R}{P_\ell}$. Let a_i be a representative of $\bar{a}_i$ in R, and b_i a representative of $\bar{b}_i$. Since $\bar{b}_i \neq 0$, we have $b_i \notin P_\ell R''$. Put $R'' = R\left[\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n}\right]$. Now let R' be the localization of R'' at the center of ν and let P'_ℓ be the l-th prime ν-ideal of R'. On the one hand,

$$P'_\ell \supset P_\ell R'; \tag{8}$$

on the other, the ring $\frac{R'}{P_\ell R'}$ is a localization of S at a prime ideal; in particular, $\frac{R'}{P_\ell R'}$ is a domain. Thus $P_\ell R'$ is prime. Now, for any element $x \in P'_\ell$, we have $x b_1^N \dots b_n^N \in P_\ell R'$ for N sufficiently large. Since $b_1 \dots b_n \notin P_\ell R'$ and $P_\ell R'$ is prime, we have $x \in P_\ell R'$. Together with (8), this shows that $P_\ell R' = P'_\ell$. The ring $\frac{R'}{P'_\ell} \cong \frac{R'}{P_\ell R'}$ is a localization of S at a prime ideal, hence it is a normal G-ring. In particular, it is analytically irreducible , as desired. $\square$

Now, take a local blowing up π such that $\kappa(P'_\ell)$ contains an extension of $\kappa(P_\ell)$ of degree d, isomorphic to some extension of $\kappa(P_\ell)$, contained in L. Applying Lemma 3.1 to the result, we obtain an R' such that, in addition, $\frac{R'}{P'_\ell} \otimes R^\dagger$ is an integral domain.

Claim. The limit in (7) is attained for this R'.

Proof of Claim: Replacing R by R', we may rephrase the Claim as follows. Assume that for any local blowing up $\pi : R \to R'$, no non-trivial extension of $\kappa(P_\ell)$, contained in $\kappa(P'_\ell)$, is isomorphic to an extension of $\kappa(P_\ell)$, contained in L. For an element $\beta \in \Phi$, let $\beta_{\ell-1} = \min\{\gamma \in \Phi \mid \beta - \gamma \in \Delta_\ell\}$. We must show that for any local blowing up π and any $\beta \in \Delta_{\ell-1} \cap \Phi$, we have

$$\mathcal{P}'_{\beta_{\ell-1}} \left(R^\dagger \otimes_R R'\right)_{M'} \bigcap R^\dagger = \mathcal{P}_{\beta_{\ell-1}} R^\dagger. \tag{9}$$

One inclusion in (9) is trivial; we must show that

$$\mathcal{P}'_{\beta_{\ell-1}} \left(R^\dagger \otimes_R R'\right)_{M'} \bigcap R^\dagger \subset \mathcal{P}_{\beta_{\ell-1}} R^\dagger. \tag{10}$$

This is the same as proving the injectivity of the map

$$\bar{\pi} : \frac{R_{P_\ell}}{\mathcal{P}_{\beta_{\ell-1}}} \otimes_R R^\dagger \to \left(\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}} \otimes_R R^\dagger\right)_T \tag{11}$$

induced by π, where T denotes the image of the multiplicative set $R' \otimes_R R^\dagger \setminus M'$ in $\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}} \otimes_R R^\dagger$ under the natural map $R' \otimes_R R^\dagger \to \frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}} \otimes_R R^\dagger$. To prove the injectivity of $\bar{\pi}$, we start with the field extension $\kappa(P_\ell) \hookrightarrow \kappa(P'_\ell)$ induced by π. Since $R^\dagger$ is flat over R, the induced map $\pi_1 : \kappa(P_\ell) \otimes_R R^\dagger \to \kappa(P'_\ell) \otimes_R R^\dagger$ is also injective. Now, $\kappa(P_\ell) \otimes_R R^\dagger$ is algebraic and finitely generated over the field $\kappa(P_\ell)$; moreover, it is a domain by the assumptions on R. Thus $\kappa(P_\ell)\otimes_R R^\dagger$ is a finite field extension of $\kappa(P_\ell)$.

Also by the assumptions on R, there is no non-trivial extension L of $\kappa(P_\ell)$ which embeds into both $\kappa(P'_\ell)$ and $\kappa(P_\ell)\otimes_R R^\dagger$. Hence the finitely generated algebraic extension

$$\kappa(P'_\ell)\otimes_R R^\dagger \cong \kappa(P'_\ell)\otimes_{\kappa(P_\ell)} R^\dagger$$

of $\kappa(P'_\ell)$ is an integral domain and hence a field. Thus $\kappa(P'_\ell)\otimes_R R^\dagger$ is a finite field extension of $\kappa(P'_\ell)$.

Next, we observe that

$$\kappa(P'_\ell)\otimes_R R^\dagger = \left(\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger\right)_{red}. \tag{12}$$

Since $\kappa(P'_\ell)\otimes_R R^\dagger$ is a field, (12) shows that $\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger$ is a local artinian ring. The multuplicative system T is disjoint from the nilradical of $\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger$, hence

$$\left(\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger\right)_T = \frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger. \tag{13}$$

Now, the map $\frac{R_{P_\ell}}{P_{\beta_{\ell-1}}}\hookrightarrow\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}$ is injective, hence so is $\frac{R_{P_\ell}}{\mathcal{P}_{\beta_{\ell-1}}}\otimes_R R^\dagger\to\frac{R'_{P'_\ell}}{\mathcal{P}'_{\beta_{\ell-1}}}\otimes_R R^\dagger$ by the flatness of $R^\dagger$ over R. Combining this with (13), we obtain (11). This completes the proof of Claim.

Let R' be a local blowing up of R which satisfies the conclusion of the Claim. Once Theorem 3.1 is proved for R', the same results for R will follow easily by intersecting all the ideals of ${R'}^\dagger$ in sight with $R^\dagger$. Therefore from now on we will replace R by R', that is, we will assume that (9) holds for any local blowing up R' of R.

We are now in the position to prove (1) of Theorem 3.1.

Let $\Phi=\nu(R\setminus\{0\})$. Since R is noetherian, Φ is well ordered. For an element $\beta\in\Phi$, let $\beta+$ denote the immediate successor of β in Φ and $\beta(\ell-1,+)$ the smallest element of Φ greater than β such that $\beta(\ell-1,+)-\beta\notin\Delta_{\ell-1}$.

Take any elements $x,y\in R^\dagger\setminus H_{2\ell-1}$. Then there exist elements $\beta,\gamma\in\Phi\cap\Delta_{\ell-1}$ such that

$$x\in P_\beta R^\dagger\setminus\mathcal{P}_{\beta(\ell-1,+)}R^\dagger \tag{14}$$

and

$$y\in P_\gamma R^\dagger\setminus\mathcal{P}_{\gamma(\ell-1,+)}R^\dagger. \tag{15}$$

To prove (1), it is sufficient to prove that

$$xy\notin\mathcal{P}_{\beta+\gamma(\ell-1,+)}R^\dagger. \tag{16}$$

Let $(a_1,...,a_n)$ be a set of generators of $\mathcal{P}_\beta$ and $(b_1,...,b_s)$ a set of generators of $\mathcal{P}_\gamma$, with $\nu(a_1)=\beta$ and $\nu(b_1)=\gamma$. Let R' be a local blowing up along ν such that R' contains all the fractions $\frac{a_i}{a_1}$ and $\frac{b_j}{b_1}$. Moreover, choose R' so that $P'_\ell{R'}^\dagger$ is a prime ideal; this is possible by Lemma 3.1. Then $a_1\mid x$ and $b_1\mid y$ in ${R'}^\dagger$. Write $x=za_1$ and $y=wb_1$ in ${R'}^\dagger$. The equality (9) implies that $z,w\notin\mathcal{P}'_\ell{R'}^\dagger$, hence

$$zw\notin\mathcal{P}'_\ell{R'}^\dagger. \tag{17}$$

We obtain $xy = a_1b_1zw$. Since ν is a valuation on R', we have $\left(\mathcal{P}'_{\beta+\gamma(l-1,+)} : (a_1b_1)R'\right) \subset P'_\ell$. By faithful flatness of $R'^\dagger$ over R' we obtain

$$\left(\mathcal{P}'_{\beta+\gamma(\ell-1,+)}R'^\dagger : (a_1b_1)R'^\dagger\right) \subset \mathcal{P}'_\ell R'^\dagger \text{ and} \tag{18}$$

$$a_1b_1 \notin \mathcal{P}_{\beta+\gamma(\ell-1,+)}R'^\dagger. \tag{19}$$

Combining this with (17), we obtain (16), as desired. This proves that the ideal $H_{2\ell-1}$ is prime. By Proposition 2.1, $H_{2\ell-1}$ maps to P_ℓ under the map Spec $R^\dagger \to$ Spec R. Since this map is étale, its fibers are zero-dimensional, which shows that $H_{2\ell-1}$ is a minimal prime of P_ℓ. This completes the proof of (1) of Theorem 3.1.

To describe the extension $\nu^\dagger$ of ν to $\frac{R^\dagger}{H_1}$, we use induction on r. For $r = 0$, R is a field, $R = R^\dagger$ (since it is a strict étale extension) and there is nothing to prove. Assume that the result is true for valuations of rank $r-1$. For an element $x \in R^\dagger \setminus H_1$, there exists an element $\beta \in \Phi$ such that $z \in \mathcal{P}_\beta R^\dagger \setminus \mathcal{P}_{\beta(0,+)}R^\dagger$ (where, of course, we allow $\beta = 0$). Let $(a_1, ..., a_n)$ be a set of generators of $\mathcal{P}_\beta$ and let $R \to R'$ be a local blowing up along ν such that R' contains all the fractions $\frac{a_i}{a_1}$ and such that $P'_1R'^\dagger$ is prime; the latter is possible by Lemma 3.1. Note that in view of (1) of Theorem 3.1, the primality of $P'_1R'^\dagger$ is also equivalent to saying that $P'_1R'^\dagger = H'_3$. Write

$$x = za_1 \tag{20}$$

in R'; we have $z \in R'^\dagger \setminus P'_1R'^\dagger$. Let $\bar\nu$ denote the rank $r-1$ valuation of $\kappa(\mathbf{m_1})$ such that $\nu = \nu_1 \circ \bar\nu$. Let $\bar\nu^\dagger$ be the unique extension of $\bar\nu$ to $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{H'_3}$. To be precise, the restriction of $\bar\nu^\dagger$ to each $\frac{R'^\dagger}{H'_3}$ is defined and is unique by the induction assumption, and $\bar\nu$ is defined and is unique by passing to the limit. Now put

$$\nu^\dagger(x) = \nu(a_1) + \bar\nu^\dagger(z). \tag{21}$$

The uniqueness of $\nu^\dagger$ is already obvious. It is also clear from (21) that every element outside of H_1 has a value in Γ, so for any prime ideal P of $R^\dagger$ such that ν extends to a valuation of $\frac{R^\dagger}{P}$, we have $P \subset H_1$, hence $P = H_1$ by the minimality of H_1.

To finish the proof of (2), it remains to show that the valuation $\nu^\dagger$ is well defined by (21). In other words, we must check that given another element $\gamma \in \Phi$ with $\gamma_0 = \beta_0$ and a factorization

$$x = wb_1 \tag{22}$$

with $w \in R'^\dagger \setminus P'_1R'^\dagger$, we have

$$\nu(a_1) + \bar\nu^\dagger(z) = \nu(b_1) + \bar\nu^\dagger(w). \tag{23}$$

Without loss of generality, we may assume that $\nu(a_1) \le \nu(b_1)$. Enlarging R', if necessary, we may assume that $b_1 = a_1t$ for some $t \in R'$. Then $tw = z$, hence $t \notin P'_1R'^\dagger$ and $\bar\nu^\dagger(z) = \bar\nu^\dagger(t) + \bar\nu^\dagger(w)$. Combining this with (20), (22) and 21), we obtain

$$\nu(b_1) + \bar\nu^\dagger(w) = \nu(a_1) + \nu(t) + \bar\nu^\dagger(w) =) = \nu(a_1) + \bar\nu^\dagger(t) + \bar\nu^\dagger(w) = \nu(a_1) + \bar\nu^\dagger(z),$$

as desired. This completes the proof of (2). (3) of Theorem 3.1 is now immediate from definitions. Theorem 3.1 is proved. □

We note the following corollary of the proof of (2) of Theorem 3.1. Let $\Phi^\dagger = \nu^\dagger(R^\dagger \setminus \{0\})$ and take an element $\beta \in \Phi^\dagger$ and let $\mathcal{P}^\dagger_\beta$ denote the $\nu^\dagger$-ideal of of $R^\dagger$ of value β.

Corollary 3.1 *Take an element $x \in \mathcal{P}^{\dagger}_{\beta}$. There exists a local blowing up $R \to R'$ such that $\beta \in \nu(R') \setminus \{0\}$ and $x \in \mathcal{P}'_{\beta} R'^{\dagger}$.*

4 The Main Theorem: the primality of the implicit ideals.

In this section we study the ideals H_j for $\hat{R}$ instead of $\tilde{R}$. The main result of this section is

Theorem 4.1 *The ideal $H_{2\ell-1}$ is prime.*

Proof: For the purposes of this proof, let $H_{2\ell-1}$ denote the implicit ideals of $\hat{R}$ and $\tilde{H}_{2\ell-1}$ the implicit prime ideals of the henselization $\tilde{R}$ of R.

Let $\tilde{R}'$ be the normalization of $\tilde{R}$. Since [6] (proposition 1, chapter IX) there exists a bijective map between the minimal ideals of $\tilde{R}$ and the maximal ideals of $tildeR'$. So $\tilde{R}'$ is a local ring [5]. Now $\tilde{R}$ is a local domain, so it's Nagata. Hence $\tilde{R}'$ is a finite $\tilde{R}$–algebra. Then $\tilde{R}'$ is noetherian [5] and the local homomorphism $\tilde{R}' \to \hat{R}'$ is regular and faithfully flat, tso $\hat{R}'$ is normal and domain. By other side, $\tilde{R}'$ is finite over $\tilde{R}$, $\hat{R}' \simeq \hat{R} \otimes_R \tilde{R}'$. Then $\hat{R}$ is a subring of $\hat{R}'$ and, in consequence, domain.

So any henselian local domain is analytically irreducible, hence $\tilde{H}_{2\ell-1}\hat{R}$ is prime for all $\ell \in \{1, \ldots, r+1\}$. Let $\tilde{\nu}$ denote the unique extension of ν to $\frac{\hat{R}}{\tilde{H}_1}$, constructed in the previous section. Let $H^*_{2\ell-1} \subset \frac{\hat{R}}{\tilde{H}_1}$ denote the implicit ideals associated to the henselian ring $\frac{\tilde{R}}{\tilde{H}_1}$ and the valuation $\tilde{\nu}$.

Claim. We have $H^*_{2\ell-1} = \frac{H_{2\ell-1}}{\tilde{H}_1}$.

Proof of Claim: For $\beta \in \Gamma$, let $\tilde{P}_\beta$ denote the $\tilde{\nu}$-ideal of $\frac{\tilde{R}}{\tilde{H}_1}$ of value β. For all β, we have $\frac{P_\beta}{\tilde{H}_1} \subset \tilde{P}_\beta$, and the same inclusion holds for all the local blowings up of R, hence $\frac{H_{2\ell-1}}{\tilde{H}_1} \subset H^*_{2\ell-1}$. To prove the opposite inclusion, we may replace $\tilde{R}$ by a finitely generated strict étale extension $R^\dagger$ of R. Now let $\Phi^\dagger = \nu^\dagger(R^\dagger \setminus \{0\})$ and take an element $\beta \in \Phi^\dagger \cap \Delta_{\ell-1}$. By Corollary 3.1, there exists a local blowing up $R \to R'$ such that $x \in P'_\beta R'^{\dagger}$. Letting β vary over $\Phi^\dagger \cap \Delta_{\ell-1}$, we obtain that if $x \in H^*_{2\ell-1}$ then $x \in \frac{H_{2\ell-1}}{\tilde{H}_1}$, as desired. This completes the proof of Claim.

The Claim shows that replacing R by $\tilde{R}$ in Theorem 4.1 does not change the problem. In other words, we may assume that R is henselian.

The rest of the proof of Theorem 4.1 follows closely the proof of Claim in the proof of Theorem 3.1. Namely, since R is a henselian G-ring, it is algebraically closed in $\hat{R}$; of course, the same holds for $\frac{R}{P_\ell}$ for all ℓ.

For an element $\beta \in \Phi$, let $\beta_{\ell-1} = \min\{\gamma \in \Phi \mid \beta - \gamma \in \Delta_\ell\}$. Let M' denote the maximal ideal of $\hat{R} \otimes_R R'$, generated by $m\hat{R} \otimes_R R$ and $\hat{R} \otimes_R m'$. First, we will show that for any local blowing up π and any $\beta \in \Delta_{\ell-1} \cap \Phi$, we have

$$P'_{\beta_{\ell-1}} \left(\hat{R} \otimes_R R'\right)_{M'} \bigcap \hat{R} = P_{\beta_{\ell-1}} \hat{R}. \tag{24}$$

One inclusion in (24) is trivial; we must show that

$$P'_{\beta_{\ell-1}} \left(\hat{R} \otimes_R R'\right)_{M'} \bigcap \hat{R} \subset P_{\beta_{\ell-1}} \hat{R}. \tag{25}$$

This is the same as proving the injectivity of the map

$$\bar{\pi} : \frac{R_{P_\ell}}{P_{\beta_{\ell-1}}} \otimes_R \hat{R} \to \left(\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}} \otimes_R \hat{R}'\right)_T \tag{26}$$

induced by π, where T denotes the image of the multiplicative set $R'\otimes_R\hat{R}\setminus M'$ in $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$ under the natural map $R'\otimes_R\hat{R}\to\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$. To prove the injectivity of $\bar{\pi}$, we start with the field extension $\kappa(P_\ell)\hookrightarrow\kappa(P'_\ell)$ induced by π. Since $\hat{R}$ is flat over R, the induced map $\pi_1:\kappa(P_\ell)\otimes_R\hat{R}\to\kappa(P'_\ell)\otimes_R\hat{R}$ is also injective. Now, since $\kappa(P_\ell)$ is algebraically closed in $\kappa(P_\ell)\otimes_R\hat{R}$ by the assumptions on R, the ring $\kappa(P'_\ell)\otimes_R\hat{R}$ is a domain.

Next, we observe that

$$\kappa(P'_\ell)\otimes_R\hat{R}=\left(\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}\right)_{red}. \tag{27}$$

Now, $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}$ is a local artinian ring and its only associated prime is its nilradical, the ideal $P'_\ell\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}$; in particular, the (0) ideal in this ring has no embedded components. Since $\hat{R}$ is flat over R, $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$ is flat over $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}$ by base change. Hence the (0) ideal of $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$ has no embedded components. In particular, since the multiplicative system T is disjoint from the nilradical of $\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$, the set T contains no zero divisors, so localization by T is injective.

Now, the map $\frac{R_{P_\ell}}{P_{\beta_{\ell-1}}}\hookrightarrow\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}$ is injective, hence so is $\frac{R_{P_\ell}}{P_{\beta_{\ell-1}}}\otimes_R\hat{R}\to\frac{R'_{P'_\ell}}{P'_{\beta_{\ell-1}}}\otimes_R\hat{R}$ by the flatness of $\hat{R}$ over R. Combining this with the injectivity of the localization by T, we obtain (26). This completes the proof of (24).

We now finish the proof of Theorem 4.1.

Let Φ be as above; we will continue to use the notation $\beta(\ell-1,+)$ for the smallest element of Φ greater than β such that $\beta(\ell-1,+)-\beta\notin\Delta_{\ell-1}$.

Take any elements $x,y\in\hat{R}\setminus H_{2\ell-1}$. Then there exist elements $\beta,\gamma\in\Phi\cap\Delta_{\ell-1}$ such that

$$x\in P_\beta\hat{R}\setminus P_{\beta(\ell-1,+)}\hat{R} \tag{28}$$

and

$$y\in P_\gamma R^\dagger\setminus P_{\gamma(\ell-1,+)}\hat{R}. \tag{29}$$

To prove (1), it is sufficient to prove that

$$xy\notin P_{\beta+\gamma(\ell-1,+)}R^\dagger. \tag{30}$$

Let $(a_1,...,a_n)$ be a set of generators of P_β and $(b_1,...,b_s)$ a set of generators of P_γ, with $\nu(a_1)=\beta$ and $\nu(b_1)=\gamma$. Let R' be a local blowing up along ν such that R' contains all the fractions $\frac{a_i}{a_1}$ and $\frac{b_j}{b_1}$. Moreover, choose R' so that $P'_\ell\hat{R}'$ is a prime ideal; this is possible by Lemma 3.1. Then $a_1\mid x$ and $b_1\mid y$ in $\hat{R}'$. Write $x=za_1$ and $y=wb_1$ in $\hat{R}'$. The equality (24) implies that $z,w\notin P'_\ell\hat{R}'$, hence

$$zw\notin P'_\ell\hat{R}'. \tag{31}$$

We obtain $xy=a_1b_1zw$. Since ν is a valuation on R', we have $\left(P'_{\beta+\gamma(\ell-1,+)}:(a_1b_1)R'\right)\subset P'_\ell$. By faithful flatness of $\hat{R}'$ over R' we obtain

$$\left(P'_{\beta+\gamma(\ell-1,+)}\hat{R}':(a_1b_1)\hat{R}'\right)\subset P'_\ell\hat{R}' \text{ and} \tag{32}$$

$$a_1b_1\notin P_{\beta+\gamma(\ell-1,+)}\hat{R}'. \tag{33}$$

Combining this with (31), we obtain (30), as desired. This completes the proof of Theorem 4.1. □

Lemma 4.1 *There exists a unique minimal prime ideal H_{2l} of $P_l\hat{R}$, contained in H_{2l+1}.*

Proof: Since $H_{2l+1} \cap R = P_l$, H_{2l+1} belongs to the fiber of the map $Spec\ \hat{R} \to Spec\ R$ over P_l. Since R was assumed to be a G-ring, $S := \hat{R} \otimes_R \kappa(P_l)$ is a regular ring. Hence its localization $\bar{S} := S_{H_{2l+1}S} \cong \frac{\hat{R}_{H_{2l+1}}}{P_l\hat{R}_{H_{2l+1}}}$ is a regular *local* ring. In particular, $\bar{S}$ is an integral domain, so (0) is its unique minimal prime ideal. The set of minimal prime ideals of $\bar{S}$ is in one-to-one correspondence with the set of minimal primes of P_l, contained in H_{2l+1}, which shows that such a minimal prime H_{2l} is unique, as desired. □

Proposition 4.1 *We have $H_{2\ell-2} \cap R = P_{\ell-1}$.*

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