PREPUBLICACIONES DEL DEPARTAMENTO DE ÁLGEBRA

DE LA UNIVERSIDAD DE SEVILLA

Explicit constructions in discrete valuations

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Prepublicación nº 3 (Febrero–2000)

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EXPLICIT CONSTRUCTIONS IN DISCRETE VALUATIONS

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ABSTRACT. In this paper we give an explicit construction of parametric equations of a discrete valuation. This amounts to find a parameter and a field of coefficients. We devote section 1 to the construction of the parameter, an element of value 1. The field of coefficients is the residue field of the valuation, explicitly constructed in section 2. At the end of this section we give the parametric equations.

TERMINOLOGY AND PRELIMINARIES

Let k be an algebraically closed field of characteristic 0, $R_n = k[\![X_1, \dots, X_n]\!]$, $M_n = (X_1, \dots, X_n)$ the maximal ideal and $K_n = k((X_1, \dots, X_n))$ the quotient field. Let v be a rank-one discrete valuation of $K_n|k$, R_v the valuation ring, $\mathfrak{m}_v$ the maximal ideal and Δ_v the residual field of v. The center of v in R_n is $\mathfrak{m}_v \cap R_n$, throughout this paper "discrete valuation of $K_n|k$" means "rank-one discrete valuation of $K_n|k$ whose center in R_n be the maximal ideal M_n". The dimension of v is the transcendence degree of Δ_v over k. We shall suppose that the group of v is $\mathbb{Z}$.

Let $\widehat{K}_n$ be the completion of K_n with respect to v, $\widehat{v}$ the extension of v to $\widehat{K}_n$, $R_{\widehat{v}}$, $\mathfrak{m}_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ the ring, maximal ideal and the residual field of $\widehat{v}$. We know that Δ_v and $\Delta_{\widehat{v}}$ are isomorphic. Let $\sigma : \Delta_{\widehat{v}} \to R_{\widehat{v}}$ be a $k-$section of the natural homomorphism $R_{\widehat{v}} \to \Delta_{\widehat{v}}$, $\theta \in R_{\widehat{v}}$ an element of value 1 and t an indeterminate. We consider the $k-$isomorphism

$$\Phi = \Phi_{\sigma,\theta} : \Delta_{\widehat{v}}[\![t]\!] \to R_{\widehat{v}}$$

given by

$$\Phi\left(\sum \alpha_i t^i\right) = \sum \sigma(\alpha_i)\theta^i,$$

and denote also by Φ its extension to the quotient fields; we have a $k-$isomorphism Φ^{-1} which composed with the order function on $\Delta_{\widehat{v}}((t))$ gives the valuation $\widehat{v}$. This is the situation we shall consider throughout all this paper, and we'll freely use it without new explicit references.

In this paper we shall use two basic transformations to construct an element of value 1 and the residual field:

1. *monoidal transformation:*

$$\begin{array}{rcl} k[\![X_1, \dots, X_n]\!] & \to & k[\![Y_1, \dots, Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_2 & \mapsto & Y_1 Y_2 \\ X_i & \mapsto & Y_i, \ i = 3, \dots, n. \end{array}$$

2. *coordinates change:*

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \to & L[\![Y_1,\dots,Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_i & \mapsto & Y_i+\sigma(b_i)Y_1,\ i=2,\dots,n, \end{array}$$

with $L\subset\sigma(\Delta_{\widehat{v}})$ is an extension field of k.

In both transformations we can see the next facts:

1. The transformations are one to one: In the case of the monoidal transformations this property is well known. In the other case it's a consequence of [5] (corollary 2, page 137).
2. The new variables $\{Y_1,\dots,Y_n\}$ are formally independent over k or L: This is a straight consequence of the previous property.
3. The new rings $k[\![Y_1,\dots,Y_n]\!]$ and $L[\![Y_1,\dots,Y_n]\!]$ are included in $R_{\widehat{v}}$, so we can use the extension $\widehat{v}$ of v to extend v to a discrete valuation v' of $k((Y_1,\dots,Y_n))$ or $L((Y_1,\dots,Y_n))$, the residual field of v' is $\Delta_{\widehat{v}}$.

Throughout this paper transformation means monoidal transformation, coordinates changes or a finite composition of these.

1. Construction of an element of value 1

In this section we shall give an effective method to construct an element of value 1 (the parameter of parametric equations).

Lemma 1.1. *Let $\alpha_i=v(X_i)$ for all $i=1,\dots,n$. By a finite number of monoidal transformations we can extend v to a new valuation v_1 of $k((Y_1,\dots,Y_n))$ such that $v_1(Y_i)=v_1(Y_1)$ for all i.*

Proof. We can suppose that $v(X_1)=\alpha_1=\min\{\alpha_i|1\le i\le n\}$ and consider two steps in the algorithm:

Step 1.- If exists $n_i\in\mathbb{Z}$ such that $\alpha_i=n_i\alpha_1$ for all $i=2,\dots,n$, then for each i we apply n_i-1 monoidal transformations like

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \to & k[\![Y_1,\dots,Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_i & \mapsto & Y_1Y_i \\ X_j & \mapsto & Y_j,\ j\neq i. \end{array}$$

Trivially $\widehat{v}(Y_i)=\alpha_1$ for all $i=1,\dots,n$.

Step 2.- If exists i, with $2\le i\le n$, such that $v(X_1)=\alpha_1$ doesn't divide to $v(X_i)=\alpha_i$ (we can suppose that $i=2$) then $\alpha_2=q\alpha_1+r$. So we apply q times the monoidal transformation

$$\begin{array}{rcl} k[\![X_1,\dots,X_n]\!] & \to & k[\![Y_1,\dots,Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_2 & \mapsto & Y_1Y_2 \\ X_i & \mapsto & Y_i,\ i=3,\dots,n \end{array}$$

to obtain a new ring $k[\![Y_1,\dots,Y_n]\!]$ where $\widehat{v}(Y_2)=r>0$ and is the variable of minimum value.

As the values of the variables are greater than zero, in a finite number of steps 2 we come to the situation of step 1. In fact, this algorithm is equivalent to "Euclidean algorithm" to compute the greatest common divisor of $\alpha_1,\dots,\alpha_n$.

□

Theorem 1.2. *We can construct an element of value 1 applying a finite number of monoidal transformations and coordinates changes.*

Proof. We can suppose that $v(X_i) = v(X_1) = \alpha$ for all $i = 2, \dots, n$ by the previous lemma. We can proof that there exists $b_i \in \Delta_{\widehat{v}}$ for each $i = 2, \dots, n$ such that $\widehat{v}(X_i - \sigma(b_i)X_1) > \alpha$: We can take

$$(\Phi_{\sigma,\theta})^{-1}(X_i) = \sum_{j \geq \alpha} a_{ij} t^j = \omega_i(t), \; a_{ij} \in \Delta_{\widehat{v}}, \; a_{i\alpha} \neq 0,$$

so $b_i = a_{i\alpha}/a_{ij}$.

Step 1.- We apply the coordinate change

$$\begin{array}{rcl} k[\![X_1, \dots, X_n]\!] & \to & L[\![Y_1, \dots, Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_i & \mapsto & Y_i + \sigma(b_i)Y_1, \; i = 2, \dots, n. \end{array}$$

With this transformation the new variables have different values.

Step 2.- We apply lemma 1.1 to even the values of variables and go to step 1. In any case, the minimum value of the variables don't increase, because this is equivalent to find the greater common divisor of the values of the variables, and the first variable does not change.

Eventually we obtain an element of value 1, then we've finished.

We have to show that the algorithm produce an element of value one in a finite number of transformations. The only way to enter in an infinite process is that, in the step 2, the minimum value of the variables doesn't decrease. This means that, in the step 1, the value of the first variable ever divides to the values of the news variables.

The composition of steps 1 and 2 is the transformation

$$\begin{array}{rcl} k[\![X_1, \dots, X_n]\!] & \to & L[\![Y_1, \dots, Y_n]\!] \\ X_1 & \mapsto & Y_1 \\ X_i & \mapsto & Y_i + \sigma(b_i)Y_1^{m_i}, \; i = 2, \dots, n. \end{array}$$

If we use the steps 1 and 2 infinitely, we have the infinite sequence of transformations

$$\begin{array}{rcl} k[\![X_1, \dots, X_n]\!] & \to & L[\![Y_1, \dots, Y_n]\!] \\ X_1 & \mapsto & Y_{1,j} \\ X_i & \mapsto & Y_{i,j} + \sum_{k=1}^{j} \sigma(b_{i,k})Y_{1,j}^{m_{i,k}}, \; i = 2, \dots, n. \end{array}$$

Then we have an infinite sequence of variables

$$\begin{array}{rcl} Y_{1,j} & = & X_1 \\ Y_{i,j} & = & X_1 - \sum_{k=1}^{j} \sigma(b_{i,k})X_1^{m_{i,k}}, \; i = 2, \dots, n, \end{array}$$

with $\widehat{v}(Y_{i,j}) > \widehat{v}(Y_{i,j-1})$ for all $i, \; j$. So any sequence of partial sums of the series

$$X_1 - \sum_{k=1}^{\infty} \sigma(b_{i,k})X_1^{m_{i,k}}, \; \forall i = 2, \dots, n.$$

have increasing values. Then these series converge to zero in $R_{\widehat{v}}$, so

$$X_1 = \sum_{k=1}^{\infty} \sigma(b_{i,k})X_1^{m_{i,k}}, \; \forall i = 2, \dots, n.$$

Let $f(X_1, \ldots, X_n) \in K_n$, then

$$v(f) = \widehat{v}\left(f\left(X_1, \sum_{k=1}^{\infty} \sigma(b_{2,k}) X_1^{m_{2,k}}, \ldots, \sum_{k=1}^{\infty} \sigma(b_{n,k}) X_1^{m_{n,k}}\right)\right) = m \cdot v(X_1).$$

In this situation, the values group of v is $v(X_1) \cdot \mathbb{Z}$ (proof in [2]) but we suppose at preliminaries that the group is $\mathbb{Z}$, so $v(X_1) = 1$. □

Example 1.3. Let us consider the embedding

$$\begin{array}{rcl} \Psi : \mathbb{C}[\![X_1, X_2, X_3]\!] & \to & \mathbb{C}[\![t, T_2, T_3]\!] \\ X_1 & \mapsto & t^2 \\ X_2 & \mapsto & T_2 t^4 + T_2 t^6 \\ X_3 & \mapsto & T_2 t^2 + T_3 t^5 \end{array}$$

with t, T_2 and T_3 variables over $\mathbb{C}$. We are going to denote Ψ to its extension to the quotient fields. The composition of this injective homomorphism with the order function in t gives a discrete valuation of $\mathbb{C}((X_1, X_2, X_3))|\mathbb{C}$, $v = \nu_t \circ \Psi$. If we apply the procedure given in this section, then we construct the next element of value 1:

$$\frac{X_3 - \sigma(b_2) X_1}{X_1^2},$$

where $b_2 \in \Delta_{\widehat{v}}$ such that $\Psi(\sigma(b_2)) = T_2$.

2. Construction of the residual field

In this section we give a finite procedure to construct the residual field Δ_v of a discrete valuation of $K_n|k$, like a transcendental extension of k. In fact, we shall extend the valuation v to other valuation v' such that $\Delta_v = \Delta_{v'}$. We want v' to be "as close as possible" to an order function.

Remark 2.1. Preliminary transformation over v.
1) We can suppose that $v(X_i) = \alpha$ for all $i = 1, \ldots, n$ by lemma 1.1
2) In this situation $v(X_2/X_1) = 0$, so $0 \neq (X_2/X_1) + \mathfrak{m}_v \in \Delta_v$. If this residue is in k then exists $a_{21} \in k$ such that

$$\frac{X_2}{X_1} + \mathfrak{m}_v = a_{21} + \mathfrak{m}_v,$$

so

$$\frac{X_2}{X_1} - a_{21} = \frac{X_2 - a_{21} X_1}{X_1} \in \mathfrak{m}_v,$$

and then

$$v\left(\frac{X_2 - a_{21} X_1}{X_1}\right) > 0.$$

So we have $v(X_2 - a_{21} X_1) = \alpha_1 > \alpha$. If α divides to α_1 then $\alpha_1 = r_1 \alpha$ with $r_1 \geq 2$ and

$$v\left(\frac{X_2 - a_{21} X_1}{X_1^{r_1}}\right) = 0.$$

If the residue of this element is too in k, then exist $a_{2r_1} \in k$ such that $v(X_2 - a_{21} X_1 - a_{2r_1} X_1^{r_1}) = \alpha_2 > \alpha_1$. If α divides to α_2 then $\alpha_2 = r_2 \alpha$ with $r_2 > r_1$ and we can repeat this operation.

3) The previous procedure is finite. If it doesn't stop we can construct the series

$$X_2 - \sum_{i=1}^{\infty} a_{2i} X_1^i$$

such that the sequence of partial sums has increasing values. So this series is zero in contradiction with X_1 and X_2 are formally independent variables.
4) We can apply this procedure to all the variables X_j $j = 2, \ldots, n$, then we have the transformations:

$$Y_1 = X_1,$$

$$Y_j = X_j - \sum_{i=1}^{s_i} a_{ji} X_1^j, \; j = 2, \ldots, n,$$

such that, in the ring $k[\![Y_1, \ldots, Y_n]\!]$, occurs one of the next two situations:
a) $v(Y_j) = v(Y_1)$ and the residue of Y_j/Y_1 is not in k, or
b) $v(Y_j) \neq v(Y_1)$ and $v(Y_1)$ does not divide to $v(Y_j)$.
In the case a), we make the transformation

$$Z_j = Y_j - \sum_{i=1}^{s-1} a_{ji} Y_1^i, \; j \neq 1,$$

with s such that Z_j/Y_1^s is the first residue which is not in k, as in the previous procedure.

In the case b) we can make the transformation

$$Z_j = Y_j - \sum_{i=1}^{s-1} a_{ji} Y_1^i, \; j \neq 1,$$

with s such that $v(Z_j)$ is the first which $v(Y_1)$ doesn't divide to $v(Y_j)$.

In any case, these transformations are the composition of monoidal transformations and coordinates changes. After these transformations we make a finite number of monoidal transformations to even the value of the variables. The new value is minor than the old $(v(X_1))$.

Anyway this procedure stops, because the value of the variables are major or equal than 1.
5) Then we can suppose that v is a discrete valuation of $K_n|k$ such that $v(X_i) = v(X_1) = \alpha$ and the residues $X_i/X_1 + \mathfrak{m}_v$ are not in k for all $i = 2, \ldots, n$. If we are not in this situation then we apply the previous procedure to do it.

The proof of the next lemma is straightforward from ([2], theorem 2.4):

Lemma 2.2. *Let v be a discrete valuation of $K_n|k$. If v is such that $v(f_r) = r\alpha$ for all form f_r of degree r respect the usual degree, then the values group of v is $\alpha \cdot \mathbb{Z}$.*

In the case of two variables we have the next

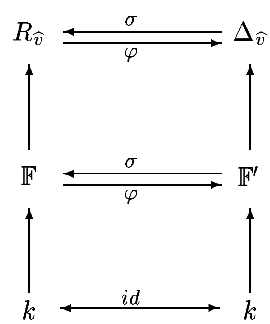
Theorem 2.3. *In the case $n = 2$, the extension of the valuation v to the quotient field of the ring constructed by the procedure of remark 2.1 is the usual order function.*

Proof. After a finite number of transformations we are in the situation of the end of remark 2.1, we note v to the extension to simplify. Let $\sigma : \Delta_{\widehat{v}} \to R_{\widehat{v}}$ a $k-$section of $R_{\widehat{v}} \to \Delta_{\widehat{v}}$, $u_2 = \sigma(X_2/X_1 + \mathfrak{m}_{\widehat{v}})$, $h \neq 0$ a form of degree r and $\gamma = X_2 - u_2 X_1$. By the construction of u_2 we know that $\widehat{v}(\gamma) > \alpha$. Then

$$h(X_1, X_2) = h(X_1, u_2X_1 + \gamma) = X_1^r h(1, u_2) + \gamma',$$

where γ' is such that $v(\gamma') > r\alpha$ (by the Newton's binomial). Like $u_2 \notin k$, then u_2 is transcendental over k, so $h(1, u_2) \neq 0$ and $v(h) = r\alpha$. By the previous lemma, the values group of v is $\alpha \cdot \mathbb{Z}$, so $\alpha = 1$ and v is the usual order function. □

Remark 2.4. We are going to construct the residual field of v as an extension of k. To do it we are going to survey all the variables searching those residues which generate the extension. Then we are going to move between $R_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ by the $k-$section σ and the natural homomorphism $\Delta_{\widehat{v}} \to R_{\widehat{v}}$. In fact we are going to construct the $k-$ section σ step by step. In this sense we have to do the next remarks:
1) Let us consider the diagram



where $\mathbb{F}$ and $\mathbb{F}'$ are subfields of $R_{\widehat{v}}$ and $\Delta_{\widehat{v}}$ respectively. Let $\omega \in R_{\widehat{v}}$ an element such that $\widehat{v}(\omega) = 0$, the question is: if $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$, is $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ transcendental over $\mathbb{F}$? What happen in the algebraic case?

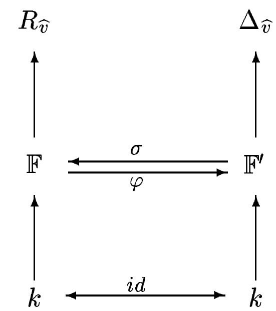
Then we suppose that $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$. Let $f(X) \in \mathbb{F}[X]$ be a non-zero polynomial. Let us put

$$f(X) = \sum_{i=0}^{n} \sigma(a'_i)X^i, \ a'_i \in \mathbb{F}'.$$

Then

$$f(\sigma(\omega + \mathfrak{m}_{\widehat{v}})) = \sum_{i=0}^{n} \sigma(a'_i)\sigma(\omega + \mathfrak{m}_{\widehat{v}})^i = \sigma\left(\sum_{i=0}^{n} a'_i(\omega + \mathfrak{m}_{\widehat{v}})^i\right) \neq 0$$

because $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$. So we've proof that $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ is transcendental over $\mathbb{F}$ if $\omega + \mathfrak{m}_{\widehat{v}}$ is transcendental over $\mathbb{F}'$
2) In the algebraic case let us consider the next diagram:



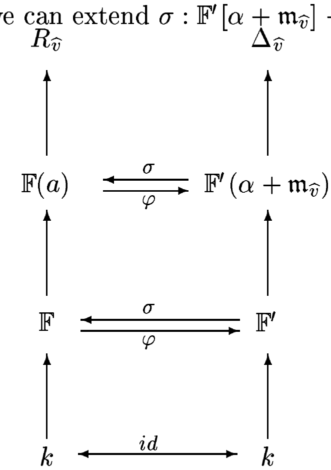
Let $\alpha + \mathfrak{m}_{\widehat{v}} \in \Delta_{\widehat{v}}$ be an algebraic element over $\mathbb{F}'$, with $\widehat{v}(\alpha) = 0$ (i.e. $\alpha + \mathfrak{m}_{\widehat{v}} \neq 0$). Let

$$\overline{f}(X) = X^m + \beta_1 X^{m-1} + \cdots + \beta_m \in \mathbb{F}'[X]$$

be its minimal polynomial over $\mathbb{F}'$. Let us take the polynomial

$$f(X) = X^m + b_1 X^{m-1} + \cdots + b_m \in \mathbb{F}[X], \text{ with } b_i = \sigma(\beta_i).$$

By the Hensel's Lemma ([5], corollary 1, page 279) we know that exists $a \in R_{\widehat{v}}$ such that a is a simple root of $f(X)$ y $\varphi(a) = \alpha + \mathfrak{m}_{\widehat{v}}$. As $\varphi\sigma = id$, $f(X)$ is the minimal polynomial of a, so we can extend $\sigma : \mathbb{F}'[\alpha + \mathfrak{m}_{\widehat{v}}] \to \mathbb{F}[a]$. Then we have



Let us consider the set

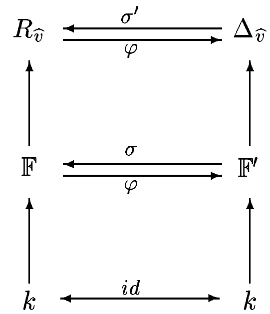
$$\Omega = \{(\mathbb{F}_1, \sigma_1) | \mathbb{F}_1 \supset \mathbb{F} \text{ and } \sigma_1 \text{ extends } \sigma\}$$

ordered by the next way

$$(\mathbb{F}_1, \sigma_1) < (\mathbb{F}_2, \sigma_2) \iff \mathbb{F}_1 \subset \mathbb{F}_2 \text{ and } \sigma_{2|\mathbb{F}_1} = \sigma_1.$$

By Zorn's lemma there exists a maximal element $(\mathbb{L}, \sigma') \in \Omega$, and again by Hensel's lemma ([5], corollary 2, page 280) we have $\varphi(\mathbb{L}) = \Delta_{\widehat{v}}$ such away. So we can extend σ to a $k-$section σ' of φ in such way that $a = \sigma'(\alpha + \mathfrak{m}_{\widehat{v}})$ is an algebraic element over $\mathbb{F}$.

3) In the next remarks we are going to give an explicit construction of the residual field of v as a subfield of $R_{\widehat{v}}$. The points **1)** and **2)** of these remark are useful to do it, because if $\omega + \mathfrak{m}_{\widehat{v}} \in \Delta_{\widehat{v}}$, $\widehat{v}(\omega) = 0$, is a transcendental (resp. algebraic) element over $\mathbb{F}'$, exists a $k-$section of φ which extends σ and $\sigma(\omega + \mathfrak{m}_{\widehat{v}})$ is transcendental (resp. algebraic) over $\mathbb{F}$.



Remark 2.5. Let us go to the general case, suppose that $n > 2$ and we have applied the procedure of remark 2.1 to construct a discrete valuation of $K_n|k$ such that

a) The value of the variables are $\alpha \in \mathbb{Z}$.

b) The residues of X_i/X_1 are transcendental over k.

1) The theorem 2.3 can be reinterpreted in the general case saying that, by a finite number of transformations, we can construct a discrete valuation v such that its restriction to $k((X_1, X_2))$ is the usual order function multiplied by α.

2) The expressions

$$\left\{ \begin{array}{c} X_1 = t^\alpha \\ X_2 = u_2 t^\alpha \end{array} \right.$$

are a parametric equations of the valuation

$$v_2 = v_{|k((X_1,X_2))}.$$

The residue field Δ_2 of v_2 is a purely transcendental extension of k of transcendence degree 1, generated by $X_2/X_1 + \mathfrak{m}_v$. $\sigma_2 : \Delta_2 \to R_{v_2}$ defined by

$$\sigma_2 \left(\frac{X_2}{X_1} + \mathfrak{m}_v \right) = \frac{X_2}{X_1}$$

is a $k-$section of the natural homomorphism. We know that exists a $k-$section σ which extends σ_2 in the sense of the previous remark.

3) Let us suppose that the residue of X_3/X_1 is algebraic over $k((X_2/X_1)+\mathfrak{m}_v)$, and let u_{31} be its image by σ. Then $v(X_3 - u_{31}X_1) = \alpha_1 > \alpha$. If α divides to α_1 then there exists $u_{3r} \in \mathrm{Im}(\sigma)$ and $r > 1$ such that $v(X_3 - u_{31}X_1 - u_{3r}X^r) = \alpha_2 > \alpha_1$. Let us suppose that u_{3r} is algebraic over $k(u_2)$ too and α divides to α_2. Then we can meet in one of the three situations shown in the next points.

4) The first situation is that, after a finite number of transformations, we obtain a value α_s such that α does not divide it. Then we make the transformation

$$Y_3 = X_3 - \sum_{j=1}^{s} u_{3j} X_1^j,$$

with u_{3j} algebraic over $k(u_2)$ for all $j = 1, \dots, s$. So we have to apply transformations to even the values of the variables and begin with all the procedure described in this section. When this situation occurs, the value of the variables decrease, so we can suppose that after a finite number of transformations we have reached an strict minimum value (it can be 1) of the variables in all this procedure. We shall denote this value by α in order not to complicate the notation. So we can suppose that this situation will never occur again for any variable.

5) The second situation is that, after a finite number of steps, we have a transcendental residue of $k(u_2)$.This means that by making the transformation

$$Y_3 = X_3 - \sum_{j=1}^{s_3} u_{3j} X_1^j,$$

and applying a finite number of monoidal transformation to equal the value of new variable we have the ring $L_3[\![X_1, X_2, Z_3, X_4, \dots, X_n]\!]$, where $L_3 = k[\{u_{3j}\}_{j=1}^{s_3}]$ and the residue $(Z_3/X_1) + \mathfrak{m}_{\widehat{v}}$ is transcendental over $k(u_2)$. We shall note $\Delta_3 = k(u_2, \{u_{3j}\}_{j=1}^{s_3}, u_3)$.
6) In this situation, if $n = 3$ then $\alpha = 1$ and the extension of v to $L((X_1, X_2, Z_3))$ is the usual order function, in analogy with the case $n = 2$ (theorem 2.3)
7) The last situation is that all the residues obtained are algebraic elements. Then we have the next theorem

Theorem 2.6. *In this situation, if all the residues are algebraic elements and $\{u_{3j}\}_{j\geq 1}$ are theirs images under σ ($k-$section constructed as in remark 2.4), then the algebraic extension*

$$k(u_2) \subset k(u_2, \{u_{3j}\}_{j\geq 1}) = \Delta_3$$

has infinite degree.

Proof. We have

$$X_3 - \sum_{j\geq 1} u_{3j} X_1^j = 0,$$

because the sequence of partial sums of this series has increasing values. Let us suppose that this extension has finite degree r. Let δ be a primitive element of the extension and $\{\delta = \delta_1, \delta_2, \dots, \delta_r\}$ the set of conjugates of δ in the minimal normal extension which contains $k(u_2, \delta)$. Let us consider the next list of elements

$$\begin{cases} \eta_1 = \varphi_0(X_1) + \varphi_1(X_1)\delta_1 + \cdots + \varphi_{r-1}(X_1)\delta_1^{r-1} \\ \qquad\vdots \\ \eta_r = \varphi_0(X_1) + \varphi_1(X_1)\delta_r + \cdots + \varphi_{r-1}(X_1)\delta_r^{r-1} \end{cases}$$

where $\varphi(X_i) \in k(u_2)((X_1))$, and η_1 is $\sum_{j\geq 1} u_{3j} X_1^j$ wrote as a linear combination of the base of the extension $\{\delta_1, \delta_1^2, \dots, \delta_1^r\}$. By general properties of Galois extension we have

$$\prod_{i=1}^{r} (Y - \eta_i) = P(Y) \in k(u_2)[\![X_1]\!][Y].$$

Then $P(X_3) = 0$, in contradiction to the fact that X_1, X_2, X_3 are formally independent. □

Remark 2.7. In all this paper we are supposing that after each transformation we come to denote by X_i to the variables, in order to simplify the notations.

Remark 2.8. Let us suppose that we have repeated the construction of remark 2.5 with each variable $X_4, \dots, X_{i-1}$, so we have a field

$$\Delta_{i-1} = k(u_2, \zeta_3, \dots, \zeta_{i-1}) \subset \sigma(\Delta_{\widehat{v}}),$$

where each ζ_k is: or $\{\{u_{kj}\}_{j=1}^{s_k}, u_k\}$ if $\{u_{kj}\}_{j=1}^{s_k}$ are algebraic elements over Δ_{k-1} and $u_k = \sigma((Z_k/X_1) + \mathfrak{m}_{\widehat{v}})$ is a transcendental element over Δ_{k-1} (i.e. the situation of remark 2.5 **5)**) , or $\{u_{kj}\}_{j\geq 1}$ if $\Delta_{k-1} \subset \Delta_k$ is an algebraic extension of infinite

degree (i.e. the situation of remark 2.5 **7)**). So we have two possible situations to the variable X_i:
1) Exists a transformation

$$Y_i = X_i - \sum_{j=1}^{s_i} u_{ij} X_1^j,$$

where the elements u_{ij} are algebraic over Δ_{i-1} and a finite number of monoidal transformations such that we have the ring $L_i[\![X_1, \ldots, X_{i-1}, Z_i, X_{i+1}, \ldots, X_n]\!]$, with $L_i = L_{i-1}(\{u_{ij}\}_{j=1}^{s_i})$ finite algebraic extension of L_{i-1} and $u_i = \sigma((Z_i/X_1) + \mathfrak{m}_{\widehat{v}})$ is a transcendental element over Δ_{i-1}. So we have the transcendental extension

$$\Delta_{i-1} \subset \Delta_{i-1}(\{u_{ij}\}_{j=1}^{s_i}, u_i) = \Delta_i.$$

2) All the elements u_{ij} which we've constructed are algebraic over Δ_{i-1}, so we have the infinite algebraic extension

$$\Delta_{i-1} \subset \Delta_{i-1}(\{u_{ij}\}_{j \geq 1}) = \Delta_i.$$

The proof of this fact is equal to the proof of theorem 2.6. In order to preserve a coherent notation we put $L_i = L_{i-1}$ in this case.

In these remarks we give an explicit construction of a ring $R_n \subset L_n[\![X_1, \ldots, X_n]\!] \subset R_{\widehat{v}}$ such that the extension of v (let us denote it again by v) to his quotient field $L_n((X_1, \ldots, X_n))$ satisfy the next properties:

1. The residual fields of these valuations, the initial and the extended, coincide because this extension is into the extension to $\widehat{v}$, so both are equal to $\Delta_{\widehat{v}}$.
2. By reordering of variables, we can suppose that the first m variables give us all the transcendental residues over k, i.e. the residue of each X_i/X_1 is transcendental over Δ_{i-1} with $i = 2, \ldots, m$. So the rest of variables $X_{m+1}, \ldots, X_n$ are such that we enter in the procedure of remark 2.5 **7)**.
3. The restriction $v_{|L_n((X_1, \ldots, X_m))}$ is the usual order function multiplied by $\alpha \in \mathbb{Z}$.
4. With the usual notations, the algebraic extension

$$\Delta_m \subset \Delta_m\left(\{u_{ij}\}_{j \geq 1}\right), \ i = m+1, \ldots, n$$

   is infinite.

So we have the next result

Theorem 2.9. *The residual field of this valuation v of $L_n((X_1, \ldots, X_n))|L_n$ (and so of the initial valuation) is*

$$\Delta_n = k\left(u_2, \{u_{3,j}\}_{j=1}^{s_3}, u_3, \ldots, \{u_{m,j}\}_{j=1}^{s_m}, u_m\right)\left(\{u_{m+1,j}\}_{j \geq 1}, \ldots, \{u_{n,j}\}_{j \geq 1}\right),$$

and the transcendence degree of Δ_n over k is $m-1$.

Proof. In this section we have given a construction to write the variables in function of X_1 and some transcendental and algebraic residues. So we have constructed the embedding

$$\begin{array}{rrcl} \varphi : L_n[\![X_1, \ldots, X_n]\!] & & \hookrightarrow & \Delta_n[\![t]\!] \\ & X_1 & \mapsto & t^\alpha \\ & X_i & \mapsto & u_i t^\alpha, \ i = 2, \ldots, m \\ & X_k & \mapsto & \sum_{j \geq \alpha} u_{k,j} t^j, \ u_{k,\alpha} \neq 0, \ k = m+1, \ldots, n. \end{array}$$

Let us denote φ its extension to the quotient field, let ν the usual order function over $\Delta_n((t))$. By the previous construction we have $v = \nu \circ \varphi$. So the residual field of v is equal to the residual field of ν, i.e. Δ_n. □

A straight consequence of this theorem is the following well-known result

Corollary 2.10. *The usual order function over K_n has dimension $n-1$, i.e. the transcendence degree of its residual field over k is $n-1$.*

Through the constructions of the remarks 2.1, 2.5 and 2.8 we arrive at a corollary which generalize the results of [2, 3]

Corollary 2.11. *Let v be a discrete valuation of $K_n|k$. The next conditions are equivalent:*

1) The transcendence degree of $\Delta_{\widehat{v}}$ over k is $n-1$.

2) Exists a finite sequence of monoidal transformations and coordinates change which transform v in an order function.

We can resume the constructions of this section in the next theorem

Theorem 2.12. *Let v be a discrete valuation of $K_n|k$, then*

1. *If the dimension of v is $n-1$, we can embed $k[\![X_1, \dots, X_n]\!]$ into a ring $L[\![Y_1, \dots, Y_n]\!]$, where $L \subset \sigma(\Delta_{\widehat{v}})$ and the extended valuation of v over the field $L((Y_1, \dots, Y_n))$ is the usual order function.*
2. *If the dimension of v is $m-1 < n-1$, we can embed $k[\![X_1, \dots, X_n]\!]$ into a ring $L[\![Y_1, \dots, Y_n]\!]$, where $L \subset \sigma(\Delta_{\widehat{v}})$ and the restriction into $L((Y_1, \dots, Y_n))$ of the extended valuation of v over $L((Y_1, \dots, Y_n))$ is the usual order function multiplied by α.*

Example 2.13. Let us consider the embedding

$$\begin{array}{rcl} \Psi : \mathbb{C}[\![X_1, X_2, X_3, X_4]\!] & \rightarrow & \mathbb{C}[\![t, T_2, T_3, T_4]\!] \\ X_1 & \mapsto & t \\ X_2 & \mapsto & T_2 t \\ X_3 & \mapsto & T_2^2 t + T_2 t^2 + T_3 t^3 \\ X_4 & \mapsto & T_2^3 t + T_2^2 t^2 + T_3 t^3 + T_4 t^4, \end{array}$$

with t, T_2, T_3 and T_4 variables over $\mathbb{C}$. We are going to denote Ψ to its extension to the quotient fields. The composition of this injective homomorphism with the order function in t gives a discrete valuation of $\mathbb{C}((X_1, X_2, X_3, X_4))|\mathbb{C}$, $v = \nu_t \circ \Psi$. The residues of X_i/X_1 are not in $\mathbb{C}$ for $i = 2, 3, 4$.

Let us put $u_2 = \sigma(X_2/X_1 + \mathfrak{m}_v)$ transcendental element over $\mathbb{C}$. By remarks 2.4 we know that we can construct σ step by step, so let take us $u_2 = X_2/X_1$ and $\Delta_2 = \mathbb{C}(u_2)$.

The residue $X_3/X_1 + \mathfrak{m}_v$ is algebraic over $\mathbb{C}(u_2)$, in fact

$$\frac{X_3}{X_1} + \mathfrak{m}_v = \frac{X_2^2}{X_1^2} + \mathfrak{m}_v.$$

So we can take $u_{31} = \sigma((X_3/X_1) + \mathfrak{m}_v) = u_2^2$. The value of $X_3 - u_{31}X_1$ is 2, the we have to see if the residue

$$\frac{X_3 - u_{31}X_1}{X_1^2} + \mathfrak{m}_v$$

is algebraic over $\mathbb{C}(u_2)$. We have that

$$\frac{X_3 - u_{31}X_1}{X_1^2} + \mathfrak{m}_v = \frac{X_2}{X_1} + \mathfrak{m}_v,$$

so it is algebraic and we can take $u_{32} = u_2$. Now $v(X_3 - u_{31}X_1 - u_{32}X_1^2) = 3$ and we have to check if

$$\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v$$

is algebraic over Δ_2, in this case, as

$$\Psi\left(\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = T_3,$$

this residue is transcendental. So we take

$$u_3 = \sigma\left(\frac{X_3 - u_{31}X_1 - u_{32}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4}.$$

Let us take $\Delta_3 = \mathbb{C}(u_2, u_3)$.

Now we have to apply this procedure to the variable X_4. The residue $X_4/X_1 + \mathfrak{m}_v$ is algebraic over Δ_3 because

$$\frac{X_4}{X_1} + \mathfrak{m}_v = \frac{X_2^3}{X_1^3} + \mathfrak{m}_v,$$

so we can take $u_{41} = \sigma((X_4/X_1) + \mathfrak{m}_v) = u_2^3 \in \Delta_3$. $v(X_4 - u_{41}X_1) = 2$, so we have to check what happens with the residue

$$\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v.$$

Now

$$\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v = \frac{X_1^2}{X_2^2} + \mathfrak{m}_v,$$

so it holds

$$u_{42} = \sigma\left(\frac{X_4 - u_{41}X_1}{X_1^2} + \mathfrak{m}_v\right) = u_2^2.$$

As $v(X_4 - u_{41}X_1 - u_{42}X_1^2) = 3$ and

$$\frac{X_4 - u_{41}X_1 - u_{42}X_1^2}{X_1^3} + \mathfrak{m}_v = \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4} + \mathfrak{m}_v,$$

we have

$$u_{43} = \sigma\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2}{X_1^3} + \mathfrak{m}_v\right) = u_3.$$

The next residue is transcendental because $v(X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3) = 4$ and

$$\Psi\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3}{X_1^4}\right) = T_4.$$

Then we can take

$$u_4 = \sigma\left(\frac{X_4 - u_{41}X_1 - u_{42}X_1^2 - u_{43}X_1^3}{X_1^4} + \mathfrak{m}_v\right) =$$

$$= \frac{X_1^2X_4 - X_3^2 - X_1^2X_2^2 - X_1^2X_3 - X_1X_2^2 - X_1^2X_2}{X_1^6}.$$

So the residual field of v is

$$\Delta_v = \mathbb{C}\left(\frac{X_2}{X_1} + \mathfrak{m}_v, \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4} + \mathfrak{m}_v, \right.$$
$$\left. \frac{X_1^2X_4 - X_3^2 - X_1^2X_2^2 - X_1^2X_3 - X_1X_2^2 - X_1^2X_2}{X_1^6} + \mathfrak{m}_v\right).$$

Then, by the transformation

$$\begin{array}{lcl} X_1 & \to & Y_1 \\ X_2 & \to & Y_2 \\ X_3 & \to & Y_1^2Y_3 + u_{3,1}Y_1 + u_{3,2}Y_1^2 \\ X_4 & \to & Y_1^3Y_4 + u_{4,1}Y_1 + u_{4,2}Y_1^2 + u_{4,3}Y_1^3, \end{array}$$

we extend the valuation v to one that is the usual order function over the field $\mathbb{C}((Y_1, Y_2, Y_3, Y_4))|\mathbb{C}$.

Remark 2.14. Exists discrete valuations of $K_n|k$ of dimension strictly less than $n-1$. [1] gives an example of a discrete valuation of $K_3|k$ of dimension 1, and [4] proves that exists discrete valuations of $K_n|k$ of dimension any number between 1 and $n-1$, this proof is constructive.

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