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The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors

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The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors

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Abstract

We find explicit free resolutions for the $\mathcal{D}$-modules $\mathcal{D}f^s$ and $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$, where f is a reduced equation of a locally quasi-homogeneous free divisor. These results are based on the fact, also proved in this paper, that every locally quasi-homogeneous free divisor is Koszul free.

Introduction

In this paper we study the module $\mathcal{D}f^s$, where $\mathcal{D}$ is the ring of germs at $0 \in \mathbb{C}^n$ of linear holomorphic differential operators and f is a reduced local equation of a locally quasi-homogeneous free divisor $D \subset (\mathbb{C}^n, 0)$.

The module $\mathcal{D}f^s$ encodes an enormous amount of geometric information of the singularity $f = 0$, but usually it is hard to work with in an explicit way. We prove the following results (see corollary 5.8 and theorem 5.9):

(A) Let $f = 0$ be a reduced local equation of a locally quasi-homogeneous free divisor of $\mathbb{C}^n$, and let $\{\delta_1, \ldots, \delta_{n-1}\}$ be a basis of the module of vector fields vanishing on f. Then

1. The δ_i generate the ideal $\operatorname{Ann}_{\mathcal{D}} f^s$.

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2. There exist explicit Koszul-Spencer type free resolutions for the modules $\mathcal{D}f^s$ and $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$ built on $\delta_1, \ldots, \delta_{n-1}$ and $f, \delta_1, \ldots, \delta_{n-1}$ respectively.

Locally quasi-homogeneous free divisors form an important class of divisors with non isolated singularities: normal crossing divisors, the union of reflecting hyperplanes of a complex reflection group, free hyperplane arrangements or the discriminant of stable mappings in Mather's "nice dimensions" are examples of such divisors.

Let X be a complex analytic manifold. Given a divisor $D \subset X$, let us write $j : U = X \setminus D \hookrightarrow X$ for the corresponding open inclusion and $\Omega^\bullet(*D)$ for the meromorphic de Rham complex with poles along D. In [11], Grothendieck proved that the canonical morphism $\Omega^\bullet(*D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism (in the derived category). This result is usually known as (a version of) *Grothendieck's Comparison Theorem*.

In [17], K. Saito introduced the *logarithmic de Rham complex* associated with D, $\Omega^\bullet_X(\log D)$, generalizing the well known case of normal crossing divisors (cf. [8]). In the same paper, K. Saito also introduced the important notion of *free divisor*.

In [7], it is proved that the logarithmic de Rham complex $\Omega^\bullet_X(\log D)$ computes the cohomology of the complement U if D is a locally quasi-homogeneous free divisor (we say that D satisfies the *logarithmic comparison theorem*). In other words, the canonical morphism $\Omega^\bullet_X(\log D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism, or using Grothendieck's result, the inclusion $\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet(*D)$ is a quasi-isomorphism. In fact, in [5] it is proved that, in the case of $\dim X = 2$, D is locally quasi-homogeneous if and only if it satisfies the logarithmic comparison theorem.

As the derived direct image $\mathbf{R}j_*(\mathbb{C}_U)$ is a perverse sheaf (it is the de Rham complex of the holonomic module of meromorphic functions with poles along D [15], II, th. 2.2.4), we deduce that $\Omega^\bullet_X(\log D)$ is perverse for every locally quasi-homogeneous free divisor.

On the other hand, the first author proved the following results [4]:
Let $D \subset X$ be a Koszul free divisor (see definition 1.6) and $\mathcal{I}$ the left ideal of the ring $\mathcal{D}_X$ of differential operators on X generated by the logarithmic vector fields with respect to D. Then
1) The left $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{I}$ is holonomic.

2) There is a canonical isomorphism in the derived category

$$\Omega_X^\bullet(\log D) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{I}, \mathcal{O}_X).$$

As a consequence, the logarithmic de Rham complex associated with a Koszul free divisor is a perverse sheaf.

The proof of **(A)** depends strongly on the following result, which has been suggested by the results above (see theorem 4.3):

(B) Every locally quasi-homogeneous free divisor is Koszul free.

In the first three sections we introduce some material concerning locally quasi-homogeneous free divisors, Koszul free divisors, the notion of ideal of linear type and the module $\mathcal{D}f^s$.

In the fourth section we prove **(B)**.

The fifth section is the main part of this paper and contains the proof of **(A)** and some related results.

In the sixth section we study some examples and we state some questions.

The first part of **(A)** has been proposed (without proof) in [1, page 240] in the particular case of discriminants of versal deformations of simple hypersurface singularities. The normal crossing divisors case has been treated in [10].

This paper is an enlarged version of our previous preprint [6].

1 Locally quasi-homogeneous and Koszul free divisors

1.1 Let X be a n-dimensional complex analytic manifold. We denote by $\pi : T^*X \to X$ the cotangent bundle, $\mathcal{O}_X$ the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of linear differential operators on X (with holomorphic coefficients), $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$ the graded ring associated with the filtration $F^\bullet$ by the order, $\sigma(P)$ the principal symbol of a differential operator P and $\{-,-\}$ the Poisson bracket on $\mathcal{O}_{T^*X}$ or $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$. We will note $\mathcal{O} = \mathcal{O}_{X,p}$, $\mathcal{D} = \mathcal{D}_{X,p}$ and $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)_p$ the respective stalks at p, with p a point of X. If $J \subset \mathcal{D}$ is a left ideal, we denote by $\sigma(J)$ the corresponding graded ideal of $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$. Given a divisor $D \subset X$, we denote by $\mathcal{D}\mathrm{er}(\log D)$ the $\mathcal{O}_X$-module of the logarithmic vector fields with respect to D [17]. If f is a local equation of D at p, we denote by $\mathrm{Der}(\log f)$ the stalk at p of $\mathcal{D}\mathrm{er}(\log D)$, whose elements are germs at p of vector fields δ such that $\delta(f) \in (f)$.

Definition 1.2.– A divisor D is Euler-homogeneous at $p \in D$ if there is a local equation h for D around p, and a germ of (logarithmic) vector field δ such that $\delta(f) = f$. A such δ is called a local Euler vector field for f.

The set of points where a divisor is Euler-homogeneous is open.

Definition 1.3.– (cf. [7]) A germ of divisor $(D,p) \subset (X,p)$ is quasi-homogeneous if there are local coordinates $x_1,\ldots,x_n \in \mathcal{O}_{X,p}$ with respect to which (D,p) has a weighted homogeneous defining equation (with strictly positive weights). A divisor D in a n-dimensional complex manifold X is locally quasi-homogeneous if the germ (D,p) is quasi-homogeneous for each point $p \in D$. A germ of divisor $(D,p) \subset (X,p)$ is locally quasi-homogeneous if the divisor D is locally quasi-homogeneous in a neighborhood of p.

Obviously a locally quasi-homogeneous divisor is Euler-homogeneous at every point.

Definition 1.4.– We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if the germ of divisor $(\{f=0\},p)$ is.

Remark 1.5.– A reduced germ $f \in \mathcal{O}_{X,p}$ is locally quasi-homogeneous if and only if for every $q \in \{f=0\}$ near p there are local coordinates $z_1,\ldots,z_n \in \mathcal{O}_{X,q}$ and a quasi-homogeneous polynomial $P(t_1,\ldots,t_n)$ (with strictly positive weights) such that $f_q = P(z_1,\ldots,z_n)$.

Definition 1.6.– ([17], [4], def. 4.1.1) Let $D \subset X$ be a divisor. We say that D is free at $p \in X$ if $\mathcal{D}\mathrm{er}(\log D)_p$ is a free $\mathcal{O}$-module (of rank n). We say that D is a Koszul free divisor at $p \in X$ if it is free at p and there exists a basis $\{\delta_1,\ldots,\delta_n\}$ of $\mathcal{D}\mathrm{er}(\log D)_p$ such that the sequence of symbols $\{\sigma(\delta_1),\ldots,\sigma(\delta_n)\}$ is regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)_p$. If D is a free (resp. Koszul free) divisor at each point of X, we simply say that it is free (resp. Koszul free). We say that a reduced germ $f \in \mathcal{O}_{X,p}$ is free if the divisor $f^{-1}(0)$ is free at p.

Let us remark that a divisor D is automatically Koszul free at every $p \in X \setminus D$.

Remark 1.7.– The ideal $I_{D,p} = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\,\mathcal{D}\mathrm{er}(\log D)_p$ is generated by the elements of any basis of $\mathcal{D}\mathrm{er}(\log D)_p$. As D is Koszul free at p if and only if $\mathrm{depth}(I_{D,p},\mathrm{Gr}_{F^\bullet}(\mathcal{D})) = n$ (cf. [14], cor. 16.8), it is clear that the definition of Koszul free divisor does not depend on the election of a particular basis. By the coherence of $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$, if a divisor is Koszul free at a point, then it is Koszul free near that point.

We have not found a reference for the following well known proposition (see [14], th. 17.4 for the local case).

Proposition 1.8.– Let $\mathbb{C}\{x\}$ be the ring of convergent power series in the variables $x = x_1, \ldots, x_n$ and let G be the graded ring of polynomials in the variables $\xi_1, \ldots, \xi_t$ with coefficients in $\mathbb{C}\{x\}$. A sequence $\sigma_1, \ldots, \sigma_s$ of homogeneous polynomials in G is regular if and only if the set of zeros $V(I)$ of the ideal I generated by $\sigma_1, \ldots, \sigma_s$ has dimension $n + t - s$ in $U \times \mathbb{C}^t$, for some open neighborhood U of 0 (then each irreducible component has dimension $n + t - s$).

Proof: Let $\mathbb{C}\{x, \xi\}$ be the ring of convergent power series in the variables $x_1, \ldots, x_n, \xi_1, \ldots, \xi_t$. As the σ_i are homogeneous and the ring $\mathbb{C}\{x, \xi\}$ is a flat extension of G, the σ_i are a regular sequence in G if and only if they are a regular sequence in $\mathbb{C}\{x, \xi\}$. But the last condition is equivalent to the equality (*loc. cit.*):

$$\dim_{(0,0)}(V(I)) = \dim\left(\mathbb{C}\{x, \xi\}/I\right) = n + t - s.$$

Finally, using the fact that all the σ_i are homogeneous in the variables ξ, the local dimension of $V(I)$ at $(0, 0)$ coincides with its dimension in $U \times \mathbb{C}^t$ for some neighborhood U of 0. C.Q.D.

Corollary 1.9.– Let $D \subset X$ be a free divisor. Let J be the ideal in $\mathcal{O}_{T^*X}$ generated by $\pi^{-1}\mathcal{D}\mathrm{er}(\log D)$. Then, D is Koszul free if and only if the set $V(J)$ of zeros of J has dimension n (in this case, each irreducible component of $V(J)$ has dimension n).

Proposition 1.10.– Let X be a complex manifold of dimension n and let $D \subset X$ be a divisor. Then:

1. Let $X' = X \times \mathbb{C}$ and $D' = D \times \mathbb{C}$. The divisor $D \subset X$ is Koszul free if and only if $D' \subset X'$ is Koszul free.

2. Let Y be another complex manifold of dimension r and let $E \subset Y$ be a divisor. Then: a) The divisor $(D \times Y) \cup (X \times E)$ is free if $D \subset X$ and $E \subset Y$ are free.

b) The divisor $(D \times Y) \cup (X \times E)$ is Koszul free if $D \subset X$ and $E \subset Y$ are Koszul free.

Proof:

1. It is a consequence of [7], lemma 2.2, (iv) and the fact that $\sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,p}[\xi_1, \ldots, \xi_n]$ if and only if $\xi_{n+1}, \sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X',(p,t)}[\xi_1, \ldots, \xi_n, \xi_{n+1}]$.

2. a) It is an immediate consequence of Saito's criterion (cf. [7], lemma 2.2, (v)).
b) It is a consequence of a) and Corollary 1.9.

C.Q.D.

Example 1.11.– Examples of Koszul free divisors are:
1) Nonsingular divisors.
2) Normal crossing divisors.
3) Plane curves: If $\dim_{\mathbb{C}} X = 2$, we know that every divisor $D \subset X$ is free [17], cor. 1.7. Let $\{\delta_1, \delta_2\}$ be a basis of $\mathcal{D}\mathrm{er}(\log D)_x$. Their symbols $\{\sigma_1, \sigma_2\}$ are obviously linearly independent over $\mathcal{O}$, and by Saito's criterion [17], 1.8, they are relatively primes in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \xi_2]$. So they form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, and D is Koszul free (see [4], cor. 4.2.2).
4) Proposition 1.10 gives a way to obtain Koszul free divisors in any dimension.
5) There are irreducible Koszul free divisors in dimensions greater than 2, which are not constructed from divisors in lower dimension [16]: $X = \mathbb{C}^3$ and $D \equiv \{f = 0\}$, with

$$f = 2^8 z^3 - 2^7 x^2 z^2 + 2^4 x^4 z + 2^4 3^2 x y^2 z - 2^2 x^3 y^2 - 3^3 y^4.$$

A basis of $\mathcal{D}\mathrm{er}(\log f)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{array}{rcccccccc} \delta_1 & = & 6y & \partial_x & + & (8z - 2x^2) & \partial_y & - & xy & \partial_z, \\ \delta_2 & = & (4x^2 - 48z) & \partial_x & + & 12xy & \partial_y & + & (9y^2 - 16xz) & \partial_z, \\ \delta_3 & = & 2x & \partial_x & + & 3y & \partial_y & + & 4z & \partial_z, \end{array}$$

and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular.

2 Ideals of linear type

Definition 2.1.– (Cf. [18], §7.2) Let A be a commutative ring, $I \subset A$ an ideal, $\mathcal{R}(I) = \oplus_{i=0}^{\infty} I^d t^d \subset A[t]$ the Rees algebra of I and $\mathrm{Sim}(I)$ the symmetric algebra of the A-module I. We say that I is of *linear type* if the canonical (surjective) morphism of graded A-algebras

$$\mathrm{Sim}(I) \to \mathcal{R}(I)$$

is an isomorphism.

Lemma 2.2.– Given a commutative ring A and an ideal $I \subset A$ generated by a family of elements $\{a_i\}_{i\in\Lambda}$, the following properties are equivalent:

a) I is of linear type.

b) If $\varphi : A[\{X_i\}_{i\in\Lambda}] \to \mathcal{R}(I)$ is the morphism of graded algebras defined by $\varphi(X_i) = a_i t$, then the kernel of φ is generated by homogeneous elements of degree 1.

Proof: We consider the kernel of the surjective morphism of graded A-algebras $\Phi : A[\{X_i\}_{i\in\Lambda}] \to \mathrm{Sim}(I)$, defined by $\Phi(X_{i_1} \cdots X_{i_d}) = a_{i_1} \cdot \ldots \cdot a_{i_d}$. Then $\ker(\Phi) = \ker(\varphi)$ if and only if I is of linear type, $\ker(\Phi)$ is an ideal generated by its homogeneous elements of degree 1, $\ker(\Phi)_1$, and $\ker(\Phi)_1 = \ker(\varphi)_1$. C.Q.D.

The definition and the lemma above sheafify in the obvious way.

The following results concern the case where the ideal I is generated by a regular sequence.

Lemma 2.3.– Let $\{a_1, \cdots, a_m\}$ be an A-sequence. For $p \le m$, if $\alpha a_1^{s_1} \cdots a_m^{s_m} \in (a_1^{s_1+k_1}, \cdots, a_p^{s_p+k_p})$, then $\alpha \in (a_1^{k_1}, \cdots, a_p^{k_p})$.

Proof: For $j = p+1, \cdots, m$, $\{a_1^{s_1+k_1}, \cdots, a_p^{s_p+k_p}, a_{p+1}^{s_{p+1}}, \cdots, a_j^{s_j}\}$ is a regular A-sequence, and we can prove inductively that

$$\alpha a_1^{s_1} \cdots a_{j-1}^{s_{j-1}} \in (a_1^{s_1+k_1}, \cdots, a_p^{s_p+k_p}).$$

For $i = p-1, \cdots, 0$, $\{a_1^{s_1+k_1}, \cdots, a_i^{s_i+k_i}, a_{i+1}^{k_{i+1}}, \cdots, a_p^{k_p}\}$ is a regular A-sequence, and we prove inductively that

$$\alpha a_1^{s_1} \cdots a_i^{s_i} \in (a_1^{s_1+k_1}, \cdots, a_i^{s_i+k_i}, a_{i+1}^{k_{i+1}}, \cdots, a_p^{k_p}).$$

C.Q.D.

Proposition 2.4.– Let A be a commutative ring and let $I \subset A$ be an ideal generated by a regular sequence $a_1, \ldots, a_n$. Then, the kernel of the morphism of graded algebras

$$A[X_1, \ldots, X_n] \to A[t], \quad X_i \mapsto a_i t,$$

is generated by $a_i X_j - a_j X_i$, $1 \leq i < j \leq n$. In particular, I is of linear type.

Proof: Let g be an homogeneous polynomial of degree m in $A[X_1, \cdots, X_n]$ such that $g(a_1, \cdots, a_n) = 0$. Let $\exp_g = cX^{e_g}$ be the greatest monomial of g in the inverse lexicographic order, with $e_g = (s_1, \cdots, s_t, 0, \cdots, 0)$, $s_t \neq 0$. Then

$$g(X_1, \cdots, X_n) - \exp_g \in (X_1^{s_1+1}, \cdots, X_t^{s_t+1}).$$

By lemma 2.3, $c = \sum_{i=1}^{t-1} \alpha_i a_i \in (a_1, \cdots, a_{t-1})$. Then

$$f(X_1, \cdots, X_n) = g(X_1, \cdots, X_n) - \sum_{i=1}^{t-1} \alpha_i (a_i X_t - a_t X_i) X_1^{s_1} \cdots X_n^{s_n}$$

is an homogeneous polynomial of degree m such that $e_f < e_g$ and

$$f(X_1, \cdots, X_n) - g(X_1, \cdots, X_n) \in J = (a_i X_j - a_j X_i, 0 < i < j \leq n).$$

In particular, $f(a_1, \cdots, a_n) = 0$. Consequently, after a finite number of steps, we will obtain $h(X_1) = c_m X_1^m$, such that $h(X_1) - g(X_1, \cdots, X_n) \in J$. So $h(a_1) = c_m a_1^m = 0$, $c_m = 0$ and $g(X_1, \cdots, X_n) \in J$.

C.Q.D.

3 The module $\mathcal{D}f^s$

Let X be a n-dimensional complex analytic manifold, p a point in X and $f \in \mathcal{O} = \mathcal{O}_{X,p}$ a non zero germ of holomophic function with $f(p) = 0$. Let D be the (germ of) divisor defined by $f = 0$. The free module of rank one over the ring $\mathcal{O}[f^{-1}, s]$ generated by the symbol f^s has a natural left module structure over the ring $\mathcal{D}[s]$ [2]: the action of a derivation $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ is given by $\delta(f^s) = \delta(f) s f^{-1} f^s$.

The following lemma is well-known and the proof is straightforward.

Lemma 3.1.– For every linear differential operator $P \in \mathcal{D}$ of order d, we have:

$$P(f^s) = C_{P,0} f^s + C_{P,1} \binom{s}{1} f^{s-1} + \cdots + C_{P,d} \binom{s}{d} f^{s-d}$$

where
$C_{P,d} = d!\sigma(P)(df) = \{\cdots\{\{\sigma(P), f\}, f\} \overset{d}{\cdots}, f\}$.

Denote by $J_f \subset \mathcal{O}$ the jacobian ideal associated with f. The surjection

$$\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}) \mapsto \delta(f) \in J_f$$

and the canonical isomorphism of graded $\mathcal{O}$-algebras

$$\mathrm{Sim}_{\mathcal{O}}(\mathrm{Der}_{\mathbb{C}}(\mathcal{O})) \simeq \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \tag{1}$$

induce a surjective graded morphism of $\mathcal{O}$-algebras

$$\varphi_f : \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \longrightarrow \mathcal{R}(J_f). \tag{2}$$

In coordinates, $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \ldots, \xi_n]$, $\xi_i = \sigma(\partial_i)$ and

$$\varphi_f(\sigma(P)) = \sigma(P)(\partial_1(f)t, \ldots, \partial_n(f)t) = \sigma(P)(df)t^d$$

for every differential operator $P \in \mathcal{D}$ of order d.

The homogeneous part of degree 1 of $\ker \varphi_f$ is naturally identified with the $\mathcal{O}$-module

$$\Theta_f = \{\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}) \mid \delta(f) = 0\}$$

by means of the canonical isomorphism (1).

Lemma 3.1 implies that $\sigma(\mathrm{Ann}_{\mathcal{D}} f^s) \subset \ker \varphi_f$.

Proposition 3.2.– With the above notations, if J_f is of linear type, then

$$\sigma(\mathrm{Ann}_{\mathcal{D}} f^s) = \ker \varphi_f$$

and the left ideal $\mathrm{Ann}_{\mathcal{D}} f^s$ of $\mathcal{D}$ is generated by Θ_f.

Proof: By lemma 2.2, $\ker \varphi_f = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\Theta_f \subset \sigma(\mathrm{Ann}_{\mathcal{D}} f^s)$.

The inclusion $\mathcal{D}\Theta_f \subset \mathrm{Ann}_{\mathcal{D}} f^s$ is obvious. Let prove that $\mathrm{Ann}_{\mathcal{D}} f^s \subset \mathcal{D}\Theta_f$. Clearly, $F^1 \mathrm{Ann}_{\mathcal{D}} f^s = \Theta_f$. Suppose $F^{d-1} \mathrm{Ann}_{\mathcal{D}} f^s \subset \mathcal{D}\Theta_f$ and take a differential operator $P \in F^d \operatorname{Ann}_{\mathcal{D}} f^s \setminus F^{d-1} \operatorname{Ann}_{\mathcal{D}} f^s$. Then, $\sigma(P) \in \ker \varphi_f = \operatorname{Gr}_{F^\bullet}(\mathcal{D})\sigma(\Theta_f)$, and $\sigma(P) = \sum A_i \sigma(\delta_i)$, where $\delta_i \in \Theta_f$ and the A_i are homogeneous of degree $d-1$. Let Q_i be differential operators such that $\sigma(Q_i) = A_i$. We apply the induction hypothesis to $P - \sum_i Q^i \delta_i \in F^{d-1} \operatorname{Ann}_{\mathcal{D}} f^s$ and we conclude the result. C.Q.D.

Proposition 3.3.– *(Isolated singularities case, cf. [13], 2.7)* If f has isolated singularity, then $\ker \varphi_f$ is generated by $\partial_i(f)\xi_j - \partial_j(f)\xi_i$, $1 \leq i < j \leq n$. In particular, the left ideal $\operatorname{Ann}_{\mathcal{D}} f^s$ of $\mathcal{D}$ is generated by $\partial_i(f)\partial_j - \partial_j(f)\partial_i$, $1 \leq i < j \leq n$.

Proof: It is a consequence of lemma 2.4 and proposition 3.2. C.Q.D.

4 Locally quasi-homogeneous free divisors are Koszul free

Proposition 4.1.– Let U be a connected open set of a complex n-dimensional analytic manifold X and let $\Sigma \subset U$ be a closed analytic set of dimension s. If a sequence $C = \{\sigma_1, \ldots, \sigma_{n-s}\}$ of homogeneous polynomials in $\mathcal{O}_X(U)[\xi_1, \ldots, \xi_n]$ is regular at every point $q \in U \setminus \Sigma$ (i.e. it is regular in $\mathcal{O}_{X,q}[\xi_1, \ldots, \xi_n]$), then it is regular at every point of U.

Proof: Let $p \in \Sigma$ and let $\pi : U \times \mathbb{C}^n \to U$ be the projection. By proposition 1.8, we have to prove that the ideal $I = (\sigma_1, \cdots, \sigma_{n-s})$ defines an analytic set $V = V(I) \subset U \times \mathbb{C}^n$ of dimension $n + s$. By hypothesis, we know that C is regular on $U \setminus \Sigma$, and so (*loc. cit.*) the dimension of (every irreducible component of) $V \cap \pi^1(U \setminus \Sigma)$ is $n + s$. Now, let W be an irreducible component of V. It has, at least, dimension $n + s$. If W is contained in $\pi^{-1}(\Sigma) = \Sigma \times \mathbb{C}^n$, then it must be equal to $\pi^{-1}(\Sigma)$. If not, $\dim W = \dim(W \cap \pi^{-1}(U \setminus \Sigma) \leq \dim(V \cap \pi^1(U \setminus \Sigma) = n + s$. So, we conclude that W has dimension $n + s$. C.Q.D.

Corollary 4.2.– Let D be a free divisor in some analytic manifold X and let $\Sigma \subset D$ a discrete set of points. If D is Koszul free at every point $x \in D \setminus \Sigma$, then D is Koszul free (at every point of X).

Theorem 4.3.– Every locally quasi-homogeneous free divisor is Koszul free.

Proof: We proceed by induction on the dimension t of the ambient manifold X. For $t = 1$, the theorem is trivial and for $t = 2$, the theorem is directly proved in examples 1.11, 3). Now, we suppose that the result is true for $t < n$, and let D be a locally quasi-homogeneous free divisor of a complex analytic manifold X of dimension n. Let $p \in D$ and let $\{\delta_1, \ldots, \delta_n\}$ be a basis of the logarithmic derivations of D at p.

Thanks to [7], prop. 2.4 and lemma 2.2, (iv), there is an open neighborhood U of p such that for each $q \in U \cap D$, with $q \neq p$, the germ of pair (X, D, q) is isomorphic to a product $(\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0,0))$, where D' is a locally quasi-homogeneous free divisor. Induction hypothesis implies that D' is a Koszul free divisor at 0. Then, by proposition 1.10.1., D is a Koszul free divisor at q too. We have then proved that D is a Koszul free divisor in $U \setminus \{p\}$. We conclude by using corollary 4.2. C.Q.D.

Corollary 4.4.– Every free divisor that is locally quasi-homogeneous at the complement of a discrete set, is Koszul free.

In particular, the last corollary gives rise a new proof of the fact that every divisor in dimension 2 is Koszul free (cf. 1.11, 3)).

5 The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors

5.1 In this section, $f \in \mathcal{O} = \mathcal{O}_{X,p}$ will be a reduced locally quasi-homogeneous free germ 1.4, 1.6. That means that $D = \{f = 0\}$ is a locally quasi-homogeneous free divisor near p.

We will also assume that

-) The equation f and its Euler vector field E are globally defined on X.

-) $E(q) \neq 0$ for every $q \in X \setminus \{p\}$.

-) $\mathcal{D}\mathrm{er}(\log D)$ is $\mathcal{O}_X$-free (of rank $n = \dim X$).

In order to proceed inductively on the dimension of the ambient variety when working with such f's, we quote the following direct consequence of [9], lemmas 1.3, 1.5 (see also [7], prop. 2.4)

Proposition 5.2.– Let $f \in \mathcal{O}_{X,p}$ a reduced locally quasi-homogeneous free germ and let D be the divisor $f = 0$. For $q \in D \setminus \{p\}$ close to p, there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ centered at q and a quasi-homogeneous polynomial $G'(t_1, \dots, t_{n-1})$ in $n-1$ variables which is also a locally quasi-homogeneous free germ in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ and such that $f_q = G'(z_1, \dots, z_{n-1})$.

We call $\widetilde{\Theta}_f$ the $\mathcal{O}_X$-sub-module (and Lie algebra) of $\mathcal{D}\mathrm{er}(\log D)$ whose sections are vector fields annhilating f. Denote by $\mathcal{J}_f \subset \mathcal{O}_X$ the jacobian ideal sheaf associated with f. The stalk of $\widetilde{\Theta}_f$ (resp. of $\mathcal{J}_f$) at p is then Θ_f (resp. J_f).

As in (2), we have a surjective graded morphism of $\mathcal{O}_X$-algebras

$$\Phi_f : \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X) \longrightarrow \mathcal{R}(\mathcal{J}_f),$$

whose stalk at p is φ_f.

We have:

$$\mathcal{D}\mathrm{er}(\log D) = \widetilde{\Theta}_f \oplus (\mathcal{O}_X E), \quad \mathrm{Der}(\log f) = \Theta_f \oplus (\mathcal{O} E), \tag{3}$$

and $\widetilde{\Theta}_f, \Theta_f$ are free of rank $n-1$.

Proposition 5.3.– The Koszul complex associated with $\widetilde{\Theta}_f \subset \mathcal{D}\mathrm{er}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^n}) = \mathcal{G}\mathrm{r}^1_{F^\bullet}(\mathcal{D}_X) \subset \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$:

$$0 \longrightarrow \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \widetilde{\Theta}_f \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_{-2}} \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{1} \widetilde{\Theta}_f \xrightarrow{d_{-1}} \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$$

$$d_{-k}(F \otimes (\sigma_1 \wedge \cdots \wedge \sigma_k)) = \sum_{i=1}^{k} (-1)^{i-1} P \sigma_i \otimes (\sigma_1 \wedge \cdots \widehat{\sigma_i} \cdots \wedge \sigma_k)$$

is exact.

Proof: We need to prove that some (or any) basis $\{\delta_1, \dots, \delta_{n-1}\}$ of $\widetilde{\Theta}_f$ form a regular sequence in $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$, but such a basis can be augmented to a basis $\{\delta_1, \dots, \delta_{n-1}, E\}$ of $\mathcal{D}\mathrm{er}(\log D)$, that we know by theorem 4.3 to form a regular sequence in $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$. C.Q.D.

Proposition 5.4.– With the hypothesis of 5.1, if the augmented graded complex of $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$-modules

$$0 \to \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \widetilde{\Theta}_f \xrightarrow{d_{-n+1}} \cdots \xrightarrow{d_{-1}} \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X) \xrightarrow{\Phi_f} \mathcal{R}(\mathcal{J}_f) \to 0 \tag{4}$$

is exact on $X - \{p\}$, then it is exact everywhere.

Proof: We know that Φ_f is surjective. By proposition 5.3, the only thing to prove is $\ker \Phi_f = \operatorname{Im} d_{-1}$. We can proceed separately on each homogeneous component:

$$0 \to \mathcal{G}\mathrm{r}^{m-n+1}_{F^\bullet}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \widetilde{\Theta}_f \xrightarrow{d^m_{-n+1}} \cdots \xrightarrow{d^m_{-1}} \mathcal{G}\mathrm{r}^m_{F^\bullet}(\mathcal{D}_X) \xrightarrow{\Phi^m_f} \mathcal{J}^m_f \to 0.$$

Let consider the coherent $\mathcal{O}_X$-module $\mathcal{F} = \operatorname{Im} d^m_{-1}$ and the short sequence

$$0 \to \mathcal{F} \to \mathcal{G}\mathrm{r}^m_{F^\bullet}(\mathcal{D}_X) \xrightarrow{\Phi^m_f} \mathcal{J}^m_f \to 0. \tag{5}$$

By proposition 5.3 and the fact that the cohomology with support $\mathcal{H}^i_p(\mathcal{O}_X)$ vanishes for $i \neq n$, we deduce that $\mathcal{H}^i_p(\mathcal{F}) = 0$ for $i = 0, 1$ and $\mathcal{H}^0_p(\mathcal{J}^m_f) = 0$. These properties and the exactness of (5) on $X - \{p\}$ imply the proposition (cf. [12], (8.14)). C.Q.D.

The following lemma is clear.

Lemma 5.5.– Let $g \in \mathcal{O}_{n-1} = \mathbb{C}\{y_1, \cdots, y_{n-1}\}$ and call $f = g$, but as an element in $\mathcal{O}_n = \mathbb{C}\{y_1, \cdots, y_n\}$. Then:

1. $\ker \varphi_f$ is generated by $\ker \varphi_g$ and $\sigma(\partial_{y_n})$.
2. Θ_f is generated by Θ_g and ∂_{y_n}.

Theorem 5.6.– Let $f \in \mathcal{O} = \mathcal{O}_{X,p}$ be a reduced locally quasi-homogeneous free germ. Then, the graded complex of $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-modules

$$0 \to \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \otimes_{\mathcal{O}} \bigwedge^{n-1} \Theta_f \overset{\varepsilon_{-n+1}}{\to} \cdots \overset{\varepsilon_{-1}}{\to} \mathrm{Gr}_{F^\bullet}(\mathcal{D}) \overset{\varphi_f}{\to} \mathcal{R}(J_f) \to 0$$

is exact. In particular, the kernel of the morphism

$$\mathrm{Gr}_{F^\bullet}(\mathcal{D}) \overset{\varphi_f}{\to} \mathcal{R}(J_f)$$

is the ideal generated by Θ_f and then the jacobian ideal J_f is of linear type.

Proof: By the exactness of (5.3), the only thing to prove is that $\ker \varphi_f$ is generated by $\sigma(\Theta_f)$. We will use induction on $n = \dim X$. If $n = 2$, we apply proposition 3.3. We suppose that the result is true if the ambient variety has dimension $n-1$. By Proposition 5.4, we need to prove the exactness of the complex (4) on $U \setminus \{x\}$, for some open neighborhood U of x, or equivalently, that $\ker \Phi_f$ is generated by $\sigma(\Theta_f)$ at every $q \in U \setminus \{x\}$. The result is then a consequence of proposition 5.2, lemma 5.5 and the induction hypothesis. C.Q.D.

Definition 5.7.– The Spencer complex[1] for $\widetilde{\Theta}_f$ is the complex of free left $\mathcal{D}_X$-modules given by:

$$0 \longrightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{n-1} \widetilde{\Theta}_f \xrightarrow{\varepsilon_{-n+1}} \cdots \xrightarrow{\varepsilon_{-2}} \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{1} \widetilde{\Theta}_f \xrightarrow{\varepsilon_{-1}} \mathcal{D}_X,$$

$$\varepsilon_{-1}(P \otimes \delta) = P\delta; \quad \varepsilon_{-k}(P \otimes (\delta_1 \wedge \cdots \wedge \delta_k)) = \sum_{i=1}^{k} (-1)^{i-1} P\delta_i \otimes (\delta_1 \wedge \cdots \widehat{\delta_i} \cdots \wedge \delta_k)$$

$$+ \sum_{1 \le i < j \le k} (-1)^{i+j} P \otimes ([\delta_i, \delta_j] \wedge \delta_1 \wedge \cdots \widehat{\delta_i} \cdots \widehat{\delta_j} \cdots \wedge \delta_k), \quad (2 \le k \le n-1).$$

In a similar way we define the Spencer complex for Θ_f, which is the stalk at p of the Spencer complex for $\widetilde{\Theta}_f$.
Both Spencer complexes can be augmented by considering the obvious maps $\mathcal{D}_X \to \mathcal{D}_X f^s, \mathcal{D} \to \mathcal{D} f^s$.

Corollary 5.8.– With the hypothesis of 5.1, we have:

(a) The Spencer complex for Θ_f is a resolution of $\mathcal{D} f^s$. In particular, the left ideal $\operatorname{Ann}_{\mathcal{D}} f^s$ is generated by Θ_f.

(b) The left ideal $\operatorname{Ann}_{\mathcal{D}[s]} f^s$ is generated by Θ_f and $E - s$.

(c) The left ideal $\operatorname{Ann}_{\mathcal{D}} \eta$, where η is the class of f^s in the quotient $\mathcal{D}[s] f^s / \mathcal{D}[s] f^{s+1}$, is generated by Θ_f and f.

Proof: For (a) we proceed as in [4], prop. 4.1.3 by using proposition 3.2 and theorem 5.6. Property (b) follows easily from (a), and property (c) follows from (a) and (b). C.Q.D.

[1] It should be noticed that such complex was originally used by Chevalley and Eilenberg in the setting of the cohomology of Lie algebras (cf. [19], 7.7).

Let us call $\Xi_f = \Theta_f \oplus (\mathcal{O}f)$ (resp. $\widetilde{\Xi}_f = \widetilde{\Theta}_f \oplus (\mathcal{O}_X f)$), which is a free sub-$\mathcal{O}$-module (respectively sub-$\mathcal{O}_X$−module) and a Lie subalgebra of $\mathcal{D}$ (resp. of $\mathcal{D}_X$). It can be also canonically embedded in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$ (resp. $\mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$) equipped with the Poisson bracket $\{-,-\}$. As in 5.3 and 5.7, we define the Koszul complex associated with $\Xi_f \subset \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ (resp. $\widetilde{\Xi}_f \subset \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$) and the Spencer complex associated with $\Xi_f \subset \mathcal{D}$ (resp. $\widetilde{\Xi}_f \subset \mathcal{D}_X$). The Koszul (resp. Spencer) complex associated with $\Xi_f \subset \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ (resp. with $\Xi_f \subset \mathcal{D}$) is obviously the stalk at p of the Koszul (resp. of the Spencer) complex associated with $\widetilde{\Xi}_f \subset \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$ (resp. with $\widetilde{\Xi}_f \subset \mathcal{D}_X$)

Theorem 5.9.– With the hypothesis of 5.1, the following properties hold:

1. The Koszul complex associated with $\Xi_f \subset \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ is exact.
2. The Spencer complex associated with $\Xi_f \subset \mathcal{D}$ is a free resolution of $\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}$.

Proof: For the first property, call $\mathbf{K}$ the Koszul complex associated with $\widetilde{\Xi}_f \subset \mathcal{G}\mathrm{r}_{F^\bullet}(\mathcal{D}_X)$. The Koszul complex associated with $\Xi_f \subset \mathrm{Gr}_{F^\bullet}(\mathcal{D})$ is the stalk at p of $\mathbf{K}$.

We proceed by induction on the dimension of the ambient variety. If that dimension is 1, $\Xi_f = \mathcal{O}f$, and the Koszul complex associated with f is clearly exact. Suppose the result true if the dimension of the ambient variety is $< n$.

Now, suppose $\dim X = n$.

Let $\delta_1, \dots, \delta_{n-1}$ a basis of $\widetilde{\Theta}_f$ in some small enough neighborhood U of p. According to proposition 4.1, we need to prove that $\mathbf{K}$ is exact on $U \setminus \{p\}$.

For every $q \in U$ with $f(q) \neq 0$, the germ of f at q is an unit and by proposition 5.3, the complex $\mathbf{K}$ is exact at q.

Let q be a point in $D = \{f = 0\}$, $q \neq p$. By proposition 5.2, there are local coordinates $z_1, \dots, z_n \in \mathcal{O}_{X,q}$ and a quasi-homogeneous polynomial $G'(t_1, \dots, t_{n-1}) \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ in $n-1$ variables which is also a locally quasi-homogeneous free germ in $\mathcal{O}_{\mathbb{C}^{n-1},0}$ and such that $f_q = G'(z_1, \dots, z_n)$.

Let $G(t_1, \dots, t_n) \in \mathcal{O}_{\mathbb{C}^n,0}$ be the same polynomial as $G'(t_1, \dots, t_{n-1})$ but considered in n variables. The exactness of $\mathbf{K}_q$ is then equivalent to the exactness of the Koszul complex associated with $\Xi_G \subset \mathrm{Gr}_{F^\bullet}\mathcal{D}_{\mathbb{C}^n,0}$.

Let us write $\mathcal{O}_m = \mathbb{C}\{t_1, \dots, t_m\}$ and call ξ_i' the principal symbol of $\frac{\partial}{\partial t_i}$.

Let $\{\delta_1', \dots, \delta_{n-2}'\} \subset \oplus_{i=1}^{n-1}\mathcal{O}_{n-1}\frac{\partial}{\partial t_i}$ be a basis of $\Theta_{G'}$. A basis of Θ_G is then $\{\delta_1', \dots, \delta_{n-2}', \frac{\partial}{\partial t_n}\} \subset \oplus_{i=1}^{n}\mathcal{O}_n\frac{\partial}{\partial t_i}$. Call σ_i' the principal symbol of $\delta_i', i = 1, \dots, n-2$.

By induction hypothesis we know that the Koszul complex associated with $\Xi_{G'} \subset \mathrm{Gr}_{F^\bullet} \mathcal{D}_{\mathbb{C}^{n-1},0} = \mathcal{O}_{n-1}[\xi'_1, \ldots, \xi'_{n-1}]$ is exact or, equivalently, that $\sigma'_1, \ldots, \sigma'_{n-2}, G'$ is a regular sequence in $\mathcal{O}_{n-1}[\xi'_1, \ldots, \xi'_{n-1}]$. That implies that $\sigma'_1, \ldots, \sigma'_{n-2}, \xi'_n, G = G'$ is a regular sequence in $\mathcal{O}_n[\xi'_1, \ldots, \xi'_n]$, i.e. that the Koszul complex associated with $\Xi_G \subset \mathrm{Gr}_{F^\bullet} \mathcal{D}_{\mathbb{C}^n,0}$ is exact, and the result is proved.

For the second property, we filter the Spencer complex associated with $\Xi_f \subset \mathcal{D}$ as in [10], prop. 2.3.4:

$$\deg(\Theta_f) = 1, \deg(f) = 0.$$

Its graded complex coincides with the Koszul complex associated with $\Xi_f \subset \mathrm{Gr}_{F^\bullet}(\mathcal{D})$, and then the Spencer complex is exact. To conclude, we use corollary 5.8, (c). C.Q.D.

6 Examples and questions

We know several (related) kind of free divisors:

[LQH] Locally quasi-homogeneous (definition 1.3).

[EH] Euler homogeneous (definition 1.2).

[LCT] Free divisors satisfying the logarithmic comparison theorem.

[KF] Koszul free (definition 1.6).

[P] Free divisors such that the complex $\Omega^\bullet_X(\log D)$ is a perverse sheaf.

We have then the following implications:

[LQH] $\Rightarrow$ [EH] (obvious), [LQH] $\Rightarrow$ [LCT] by [7], th. 1.1,
[LCT] $\Rightarrow$ [P], by [15], II, th. 2.2.4) [KF] $\Rightarrow$ [P] by [4], th. 4.2.1,
[LQH] $\Rightarrow$ [KF] by theorem 4.3.

Example 6.1.– (Free divisors in dimension 2) We recall theorem 3.9 from [5]: Let X be a complex analytic manifold of dimension 2 and $D \subset X$ a divisor. The following conditions are equivalent:
1. D is Euler homogeneous.

2. D is locally quasi-homogeneous.
3. The logarithmic comparison theorem holds for D.
Consequently, in dimension 2 we have:

$$[\mathrm{LQH}] \Leftrightarrow [\mathrm{EH}] \Leftrightarrow [\mathrm{LCT}]$$

and [KF] (cf. 1.11, 3) and [P] ([4]) always hold. In particular,

$$[\mathrm{KF}] \not\Rightarrow [\mathrm{LQH}], [\mathrm{EH}], [\mathrm{LCT}].$$

Examples of plane curves not satisfying logarithmic comparison theorem are, for instance, the curves of the family (cf. [5]):

$$x^q + y^q + xy^{p-1} = 0, \quad p \geq q+1 \geq 5.$$

Example 6.2.– (An example in dimension 3) Let consider $X = \mathbb{C}^3$ and $D = \{f = 0\}$, with $f = xy(x+y)(x+yz)$ [4]. A basis of $\mathcal{D}\mathrm{er}(\log D)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{array}{rcrcrcl} \delta_1 & = & xy\partial_x & + & y^2\partial_y & - & 4(x+yz)\partial_z, \\ \delta_2 & = & x(x+3y)\partial_x & - & y(3x+y)\partial_y & + & 4x(z-1)\partial_z, \\ \delta_3 & = & x\partial_x & + & y\partial_y & & \end{array}$$

the determinant of the coefficients matrix being $-16f$ and

$$\delta_1(f) = 0, \quad \delta_2(f) = 0, \quad \delta_3(f) = 4f.$$

In particular, D is Euler homogeneous ($E = 1/4\delta_3$) and we know [5] that it satisfies the logarithmic comparison theorem. Let $I \subset \mathcal{O}_{T^*X}$ be the ideal generated by the symbols $\{\sigma_1, \sigma_2, \sigma_3\}$ of the basis of $\mathcal{D}\mathrm{er}(\log D)$. By corollary 1.9, D is not Koszul free, because the dimension of $V(I)$ at $((0,0,\lambda),0) \in T^*X$ is 4, and D is not locally quasi homogeneous neither.

So:

$$[\mathrm{LCT}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}], \qquad [\mathrm{EH}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}].$$

Finally, for the only missing relation, we quote the following conjecture from [5]:

Conjecture 6.3.– If the logarithmic comparison theorem holds for D, then D is Euler homogeneous.

Example 6.4.– Let see that, in the example 6.2, the left ideal $\mathrm{Ann}_{\mathcal{D}}(f^s)$ is not generated by Θ_f and then, J_f is not an ideal of linear type.
Here, we set $X = \mathbb{C}^3$, $p = (0,0,0)$ and $E = 1/4\delta_3$. The $\mathcal{O}$-modules Θ_f and $\mathrm{Der}(\log f) = \Theta_f \oplus \mathcal{O} \cdot E$ are generated by $\{\delta_1, \delta_2\}$ and $\{\delta_1, \delta_2, E\}$ respectively. The symbols $\sigma_1 = \sigma(\delta_1)$, $\sigma_2 = \sigma(\delta_2)$ form a $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular sequence (the proof is analogous to example 1.11, 3)). Then, as in the proof of [4], prop. 4.1.2, we have

$$\sigma(\mathcal{D}\Theta_f) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\sigma(\Theta_f).$$

For

$$P = 2y^2\partial_x\partial_y - 2y^2\partial_y^2 - 2xz\partial_x\partial_z - 6yz\partial_x\partial_z + 10yz\partial_y\partial_z - 8z^2\partial_z^2 + 2x\partial_y\partial_z - 4y\partial_y\partial_z + 8z\partial_z^2 - x\partial_x - y\partial_y - 8z\partial_z + 4\partial_z$$

and $R = \mathbb{C}[x,y,z]$, $S = R[\xi_1, \xi_2, \xi_3]$, $\mathfrak{m} = R(x,y,z)$ we check that

1. $P \in \mathrm{Ann}_{\mathcal{D}_X}(f^s)$,
2. $(S(\sigma_1, \sigma_2) : \sigma(P)) = S(x,y)$, and then $(S(\sigma_1, \sigma_2) : \sigma(P)) \cap R = R(x,y)$.

So, $\sigma(P) \notin R_{\mathfrak{m}}[\xi_1, \xi_2, \xi_3]\sigma(\Theta_f)$ and, by faithfully flatness,

$$\sigma(P) \notin \mathcal{O}[\xi_1, \xi_2, \xi_3]\sigma(\Theta_f) = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\sigma(\Theta_f).$$

We conclude that $P \notin \mathcal{D}\Theta_f$.

Problem 6.5.– We do not know whether a free divisor defined by a quasi-homogeneous polynomial (with strictly positive weights) is locally quasi-homogeneous.

Problem 6.6.– We do not know any example of a free divisor $D \subset X$ whose logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ is not perverse.

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