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ON THE EQUATIONS DEFINING TORIC PROJECTIVE VARIETIES

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1 INTRODUCTION

Several recent results ([1], [2], [3], [4]) study the syzygies of toric varieties. In particular, the equations defining an embedded affine toric variety can be described. When the toric variety is projective one, the situation becomes special, since the semigroup defining it has a system of generators which lies on an hyperplane (i.e. there exists a map L with the properties in section 1 below).

The purpose of this paper is to give an estimation for the degrees of the equations defining an embedded projective toric variety, and give an effective upper bound for such degrees. Our results show such upper bound can be derived from some general facts stablished in [2]. As an illustration, we compute the bound explicitly in the case of toric projective curves.

2 THE APERY SET

Let S be a cancellative finitely generated commutative semigroup with zero element and torsion free. Let Λ be a finite set of generators for S, $\sharp\Lambda = h$. Denote $G(S)$ the smallest group containing S and let d be its rank. Then $G(S) \simeq \mathbf{Z}^d \subset \mathbf{Q}^d \simeq V(S) := G(S) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let $C(S)$ be the cone generated by S, i.e., the rational cone in $V(S)$ generated by the image $\overline{S}$ of S in $V(S)$.

Suppose that there exists $L : S \to \mathbf{N}$ satisfying

1. $L(\Lambda) = \{1\}$
2. $L(n+m) = L(n) + L(m)$
3. $L(n) = 0 \Longleftrightarrow n = 0$.

Fix, from now on, the data S, Λ, and L. Notice that the existence of L implies that S satisfies the property $S \cap (-S) = \{0\}$ therefore, the cone $C(S)$ is strongly convex. Denote f the number of extremal rays of $C(S)$. Notice that, since $C(S)$ generates $V(S)$, one obviously has $f \geq d$. Then, there are subsets $E \subset \Lambda$ with $\sharp E = f$ such that $C(E) = C(S)$, where $C(E)$ is the cone in $V(S)$ generated by E.

Following [2], fix a partition $\Lambda = E \cup A$, where E satisfies the above property. This kind of partition is called *convex partition*. Let $\sharp A = r = h - f$.

The Apery set Q of S relative to E is defined to be the set given by

$$Q := \{q \in S \quad | \quad q - e \notin S, \forall e \in E\}.$$

The Apery set Q is finite as proved in Proposition 5.1 of [2]. The terminology Apery comes from the use of the set Q, for the particular case of numerical semigroups, done in [5].

This finiteness property follows from the fact that the semigroup ring $\mathbf{Z}[S]$ is a finite integral extension of $\mathbf{Z}[E]$, however we will show below an effective proof of it. Set $E = \{e_1, \ldots, e_f\}$ and $A = \{a_1, \ldots, a_r\}$. We know that E contains a basis of $V(S)$ as $\mathbf{Q}$-vector space. Since $a_j \in C(S)$, for any j, we have that

$$a_j = \sum_{i=1}^{f} \lambda_{ij} e_i, \text{ with } \lambda_{ij} \in \mathbf{Q}^+.$$

Therefore, for any j, $\exists q_j \in \mathbf{N}$ such that

$$q_j a_j = \sum_{i=1}^{f} t_{ij} e_i, \text{ with } t_{ij} \in \mathbf{N}.$$

If $m \in Q$, $m = \sum_{j=1}^{r} \beta_j a_j$ with $\beta_j < q_j$, for any j. Therefore, there exists only a finite number of $m \in Q$.

REMARK 2.1 In order to find the set Q it is enough:

1. Compute q_j, for any j, $1 \leq j \leq r$.
2. Check whether the elements $m = \sum_{j=1}^{r} \beta_j a_j$, with $\beta_j < q_j$ for any j, is in Q.

Now, for any $t \geq 0$, let $H^t := \{m \in S \quad | \quad L(m) = t\}$, and denote

$$Q^t := Q \cap H^t,$$

and

$$t_0 := min\{t \quad | \quad Q^t = \emptyset\}.$$

PROPOSITION 2.2 With the above notation, if $Q^t = \emptyset$ then $Q^{t'} = \emptyset$ for all $t' \geq t$. In particular, one has $Q^t = \emptyset$ for $t \geq t_0$.

Proof. It is enough to prove that

$$Q^t = \emptyset \Rightarrow Q^{t+1} = \emptyset.$$

Suppose that $m \in Q^{t+1}$. Then, $m = m' + a$ with $a \in A$ and $m' \in H^t$. Since $Q^t = \emptyset$, we have that $m' \notin Q$ and therefore there exists $e \in E$ such that $m' - e \in S$. But then $m - e \in S$, a contradiction with $m \in Q$. □

REMARK 2.3 In order to find t_0 it is enough to use 2.1. Note that 2 in 2.1 could be computed, in practice, by using an integer programming method.

3 SOME HOMOLOGY EXACT SEQUENCES

Fix a field k for coefficients. Denote by Σ the simplex of parts of Λ. For any simplicial subcomplex Δ of Σ, let us denote by $\widetilde{H}_l(\Delta)$, $-1 \leq l \leq h-2$, the $l-th$ vector space of augmented homology of Δ with values in k, and by $\widetilde{h}_l(\Delta)$ its dimension. Note that the value $\widetilde{h}_l(\Delta)$ depends on the characteristic of the field k, however for $l = 0$ this is not so as $\widetilde{h}_0(\Delta)$ is exactly the number of connected components of Δ minus 1.

Now, for any $m \in S$, consider the simplicial subcomplexes of Σ:

$$\Delta_m = \{F \subset \Lambda \quad | \quad m - n_F \in S\}$$

and

$$T_m = \{F \subset E \quad | \quad m - n_F \in S\},$$

where $n_F := \sum_{n \in F} n$ and $n_\emptyset = 0$. Our objective will be to obtain some information of Δ_m by means of T_m. Notice that if $A = \emptyset$, then $\Delta_m = T_m$. To have a significative discussion from now on, assume that $\sharp A = r \geq 1$.

DEFINITION 3.1 On the elements of S, define a partial order $>_Q$

$$m >_Q m' \Longleftrightarrow m - m' \in S \setminus Q.$$

If $H \subset S$, we say that $m \in H$ is Q-minimal in the set H if $m \geq_Q m'$, with $m' \in H$, implies that $m = m'$.

PROPOSITION 3.2 The set

$$D(0) := \{m \in S \mid \widetilde{H}_0(T_m) \neq 0\}$$

is finite and it can be determined by an algorithm.

Proof. The finiteness of $D(0)$ is shown in Proposition 4.1 of [2]. We will give here an effective proof of this fact.

As above, set $E = \{e_1, \dots, e_f\}$ and $A = \{a_1, \dots, a_r\}$. Notice that if $\widetilde{H}_0(T_m) \neq 0$ then, by choosing elements e_1 and e_2 in different connected component of T_m, one obtains, for $j = 1, 2$:

$$m - e_j \in S \Longrightarrow m = \sum_{i=1}^{f} \alpha_i^{(j)} e_i + \sum_{i=f+1}^{f+r} \alpha_i^{(j)} a_{i-f}, \text{ with } \alpha_1^{(j)} \geq 1.$$

Set $\alpha^{(j)} = (\alpha_1^{(j)}, \dots, \alpha_h^{(j)}) \in \mathbf{N}^h$, for $j = 1, 2$, and let $\epsilon_j \in \mathbf{N}^h$, the vector with coordinates equal to zero, excepting the jth one which is equal to 1. Then, $\alpha^{(j)} \gg \epsilon_j$, for $j = 1, 2$, where $\gg$ stands for the componentwise partial order.

Sea $\mathcal{A}$ the matrix whose column vectors are the generators of S. Notice that $m = \mathcal{A}\alpha^{(j)}$, for $j = 1, 2$, and $\alpha = (\alpha^{(1)}, \alpha^{(2)}) \in \mathbf{N}^{2h}$ satisfies

$$(\mathcal{A}| - \mathcal{A})\alpha = 0, \text{ and } \alpha \gg (\epsilon_1, \epsilon_2).$$

Then, if one considers the sets

$$R_{e_1e_2} := \{\beta = (\beta^{(1)}, \beta^{(2)}) \in \mathbf{N}^{2h} \mid (\mathcal{A}| - \mathcal{A})\beta = 0, \text{ with } \beta \gg (\epsilon_1, \epsilon_2)\},$$

and

$$\Sigma R_{e_1e_2} := \{m' \in S \mid m' = \mathcal{A}\beta^{(1)} \text{ with } \beta \in R_{e_1e_2}\},$$

we have that $\alpha \in R_{e_1e_2}$ and $m \in \Sigma R_{e_1e_2}$. Furthermore, in fact one has, $m \in M_{e_1e_2}$ where

$$M_{e_1e_2} := \{m' \in \Sigma R_{e_1e_2} \mid m' \text{ is } Q-\text{minimal in } \Sigma R_{e_1e_2}\}.$$

In fact, notice that on the semigroup S one can write $m = m' + m''$, with $m' \in M_{e_1e_2}$ and $m'' \in S \setminus Q$. If $m'' \neq 0$, then one has $m'' = \sum_{j=1}^{f} \beta_j e_j + \sum_{j=f+1}^{f+r} \beta_j a_{j-f}$, $\beta_j \in \mathbf{N}$ and $\beta_j \neq 0$ for some j, $1 \leq j \leq f$, which is a contradiction since e_1 and e_2 are in different connected component of T_m.

Thus, to show that $D(0)$ is finite, it is enough to prove that each set $M_{e_1e_2}$ is so. For it, notice that the set

$$\mathcal{H}R_{e_1e_2} := \{\beta \in R_{e_1e_2} \mid \beta \text{ is minimal for } \gg\},$$

is finite and set

$$\Sigma\mathcal{H}R_{e_1e_2} = \{m' \in S \mid m' = \mathcal{A}\beta^{(1)} \text{ with } \beta \in \mathcal{H}R_{e_1e_2}\}.$$

We claim that

$$M_{e_1e_2} \subset \Sigma\mathcal{H}R_{e_1e_2} + Q.$$

The proof of the proposition follows from the claim.

In fact, let $n \in M_{e_1e_2}$. Set $n = \mathcal{A}\beta^{(1)}$, with $\beta \in R_{e_1e_2}$. If $\beta \in \mathcal{H}R_{e_1e_2}$, we claim is obvious. Otherwise, $\beta = \gamma + \mu$, with $\gamma \in \mathcal{H}R_{e_1e_2}$ and $\mu \in \mathbf{N}^{2h}$. Then, $n = n' + n''$, with $n' = \mathcal{A}\gamma^{(1)}$ and $n'' = \mathcal{A}\mu^{(1)}$. It is clear by definition that $n' \in \Sigma\mathcal{H}R_{e_1e_2}$, and $n'' \in Q$ because n is Q-minimal in $M_{e_1e_2}$. This proves the claim.

□

Above effective proof gives rise to the algorithm mentioned in the statement of the proposition. Next remark points out how one can proceed to the computation of $D(0)$.

REMARK 3.3 In order to find the set $D(0)$ it is enough:

1. Compute the set Q (see 2.1).
2. Compute the sets $\Sigma\mathcal{H}R_{e_1e_2}$, for any $e_1, e_2 \in E$, with $e_1 \neq e_2$ using integer programming (see [6]).
3. Check the Q-minimal elements in $\Sigma\mathcal{H}R_{e_1e_2} + Q$ and obtain $M_{e_1e_2}$, for any $e_1, e_2 \in E$, with $e_1 \neq e_2$.
4. Check the elements
$$m \in \bigcup_{e_1,e_2} M_{e_1e_2},$$
such that $\widetilde{H}_0(T_m) \neq 0$ by using linear algebra.

Now, let us recover from [2] a construction which shows how the set $D(0)$ can be combinatorially described.

- For any $m \in G(S)$ and $l \geq -1$, denote by $C_l(\mathbf{Q}_m)$ the vector space which has the set
$$\{L \subset A \quad | \quad \sharp L = l+1, m - n_L \in Q\}$$
as a basis.
- For any chain z in $C_t(\mathbf{Q}_m)$, denote by $\theta_t(z)$ the projection on $C_{l-1}(\mathbf{Q}_m)$ of the simplicial boundary of z.

By Lemma 2.2 in [2], $\{C_\bullet(\mathbf{Q}_m), \theta_\bullet\}$ is a chain complex for any m. To understand better the homology of this complex, consider, for any $m \in S$, the following subset of Σ:

$$\mathbf{K}_m = \{L \in \Delta_m \quad | \quad (L \cap E \neq \emptyset) \ \ or \ \ (L \subset A \ \text{ and } \ m - n_L \in S - Q)\}.$$

It is easy to check that $\mathbf{K}_m$ is a simplicial subcomplex of Δ_m, so that one can consider the chain complex $\widetilde{C}_\bullet(\mathbf{K}_m)$ and the relative chain complex $\widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$.

Notice that, by construction, one has an identification $C_\bullet(\mathbf{Q}_m) \simeq \widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$. Notice that if $m \in Q$, then one has that $C_\bullet(\mathbf{Q}_m) \simeq k$ and $\mathbf{K}_m = \emptyset$. Otherwise, if $m \in S \setminus Q$, since $\exists e \in E$ such that $m - e \in S$, one obtains $L = \{e\} \in \mathbf{K}_m$ and $\mathbf{K}_m \neq \{\emptyset\}$.

Therefore, $\widetilde{H}_{-1}(\mathbf{K}_m) = 0$ for any $m \in S$. This allows to deduce, from the exact sequence of complexes,

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\Delta_m) \to C_\bullet(\mathbf{Q}_m) \to 0,$$

that there is a long exact sequence of homology,

$$... \to H_{l+1}(\mathbf{Q}_m) \to \widetilde{H}_l(\mathbf{K}_m) \to \widetilde{H}_l(\Delta_m) \to H_l(\mathbf{Q}_m) \to ...$$
$$... \to \widetilde{H}_0(\mathbf{K}_m) \to \widetilde{H}_0(\Delta_m) \to H_0(\mathbf{Q}_m) \to \widetilde{H}_{-1}(\mathbf{K}_m) = 0 \to$$
$$\widetilde{H}_{-1}(\Delta_m) \to H_{-1}(\mathbf{Q}_m) \to 0.$$

Now, in order to understand the homology $\widetilde{H}_\bullet(\mathbf{K}_m)$, let us consider the simplicial complex given by the following disjoint union of subsets of Σ:

$$\overline{\mathbf{K}}_m := \mathbf{K}_m \cup \{I \cup J, I \subset A, J \subset E \quad | \quad m - n_I - n_J \notin S \ \text{ and } \ m - n_I - e \in S, \forall e \in J\}.$$

Notice that any $I \cup J$ in the second set of the above union is such that the cardinality of J is at least 2. The complex $\overline{\mathbf{K}}_m$ is acyclic, i.e. $\widetilde{H}_l(\overline{\mathbf{K}}_m) = 0$ for any $l \geq -1$ (see Corollary 2.1 in [2]). Thus, the long exact sequence of homology coming from the exact sequence of chain complexes

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to 0$$

gives rise to an isomorphism $\rho_{l+1} : \widetilde{H}_{l+1}(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_l(\mathbf{K}_m)$, for every $l \geq -1$.

To study the homology $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ let us consider, the chain of simplicial complexes

$$\mathbf{K}_m = \mathbf{M}_m^{(-1)} \subset \mathbf{M}_m^{(0)} \subset \mathbf{M}_m^{(1)} \subset ... \subset \mathbf{M}_m^{(r)} = \overline{\mathbf{K}}_m$$

where $\mathbf{M}_m^{(i)}$, $-1 \leq i \leq r$, is the simplicial subcomplex of $\overline{\mathbf{K}}_m$ given by:

$$\mathbf{M}_m^{(i)} := \mathbf{K}_m \cup \{L = I \cup J \in \overline{\mathbf{K}}_m \quad | \quad I \subset A, J \subset E, \text{ and } \sharp I \leq i\}.$$

Now, $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ can be computed (see [2]) by means of the long exact sequences

$$... \to \widetilde{H}_l(\mathbf{M}_m^{(j)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(j)}) \to ....$$

for $-1 \leq i < j < k \leq r$. In fact, to compute $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) = \widetilde{H}_\bullet(\mathbf{M}_m^{(r)}, \mathbf{M}_m^{(-1)})$, it will be enough to use the above exact sequences for the concrete values of (i, j, k) given by $(-1, 0, 1), (-1, 1, 2), ..., (-1, r-1, r)$, and take into account the following result which is obvious by construction (see proposition 4.3 in [2]).

PROPOSITION 3.4 With the previous notations, for any $m \in S$, one has: for any $l \geq -1$ and any i, $0 \leq i \leq r$,

$$\widetilde{H}_{l+1}(\mathbf{M}_m^{(i)}, \mathbf{M}_m^{(i-1)}) \simeq \bigoplus_{I \subset A, \sharp I = i} \widetilde{H}_{l-i}(T_{m-n_I})$$

(in this formula, $\widetilde{H}_{l-i}(T_{m-n_I}) = 0$ if either $l + 1 < i$ or $m - n_I \notin S$).

4 DEGREES OF THE EQUATIONS

It is known that the degrees of the elements in a minimal generating set of the ideal of S are exactly

$$\{L(m) \mid \widetilde{h}_0(\Delta_m) \neq 0\}$$

(see [7] and [4]). Moreover, each such degree t appears as many times as the sum of the values $\widetilde{h}_0(\Delta_m)$ for $m \in H^t$. In order to estimate these degrees, we are going to use the formulae and exact sequences in above section and, in particular, the following four terms one

$$... \to H_1(\mathbf{Q}_m) \to \widetilde{H}_1(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_0(\Delta_m) \to H_0(\mathbf{Q}_m) \to 0.$$

PROPOSITION 4.1 Let $m \in S \cap H^t$ with $t > t_0$, then $H_0(\mathbf{Q}_m) = 0$.

Proof. If $m-a \in Q$ and $m \in S \cap H^t$, then $m-a \in Q^{t-1}$. Since $t > t_0$, $t-1 \geq t_0$ and so $Q^{t-1} = \emptyset$ by 2.2. Thus, one has that $C_0(\mathbf{Q}_m) = \bigoplus_{m-a \in Q} k\{a\} = 0$ and $H_0(\mathbf{Q}_m) = 0$. $\square$

Now, on the other hand, by 3.4 one has for $l = 0$

$$\widetilde{H}_1(\mathbf{M}_m^{(i)}, \mathbf{M}_m^{(i-1)}) \simeq \bigoplus_{I \subset A, \sharp I = i} \widetilde{H}_{-i}(T_{m-n_I}).$$

Thus, one has

$$\widetilde{H}_1(\mathbf{M}_m^{(i)}, \mathbf{M}_m^{(i-1)}) \simeq \bigoplus_{I \subset A, \sharp I = i} \widetilde{H}_{-i}(T_{m-n_I}) = 0,$$

for $i > 1$, and the following diagram of exact sequences

$$... \to \widetilde{H}_1(\mathbf{M}_m^{(0)}, \mathbf{M}_m^{(-1)}) \to \widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(-1)}) \to \widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(0)}) \to ....$$

$$... \to \widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(-1)}) \to \widetilde{H}_1(\mathbf{M}_m^{(2)}, \mathbf{M}_m^{(-1)}) \to \widetilde{H}_1(\mathbf{M}_m^{(2)}, \mathbf{M}_m^{(1)}) = 0 \to ...$$

$$\vdots$$

$$... \to \widetilde{H}_1(\mathbf{M}_m^{(r-1)}, \mathbf{M}_m^{(-1)}) \to \widetilde{H}_1(\mathbf{M}_m^{(r)}, \mathbf{M}_m^{(-1)}) = \widetilde{H}_1(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_1(\mathbf{M}_m^{(r)}, \mathbf{M}_m^{(r-1)}) = 0 \to ....$$

Again by 3.4 one has:

For $l = 0$, $i = 1$:

$$\widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(0)}) \simeq \bigoplus_{a \in A} \widetilde{H}_{-1}(T_{m-a}).$$

For $l = 1$, $i = 1$:

$$\widetilde{H}_2(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(0)}) \simeq \bigoplus_{a \in A} \widetilde{H}_0(T_{m-a}).$$

For $l = 0$, $i = 0$:

$$\widetilde{H}_1(\mathbf{M}_m^{(0)}, \mathbf{M}_m^{(-1)}) \simeq \widetilde{H}_0(T_m).$$

Replacing in the first sequence of the above diagram, one obtains

$$.. \to \bigoplus_{a \in A} \widetilde{H}_0(T_{m-a}) \overset{\varphi_m}{\to} \widetilde{H}_0(T_m) \to \widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(-1)}) \to \bigoplus_{a \in A} \widetilde{H}_{-1}(T_{m-a}) \to ...$$

REMARK 4.2 In order to describe the mapping φ_m we can do the following:

1. Compute the simplicial complexes T_m and T_{m-a} with $a \in A$. (Using Integer Programming)
2. Take bases of $\widetilde{H}_0(T_m)$ and $\widetilde{H}_0(T_{m-a})$ with $a \in A$, picking a point $\{e\}$ in each connected component and considering a generating tree, for example fix $\{e_1\}$ for one concrete of the components and consider $\{e_1\} - \{e_i\}$, with e_i over the other components (see [7] for details).
3. Take the natural basis of $\bigoplus_{a \in A} \widetilde{H}_0(T_{m-a})$ obtained from the bases of $\widetilde{H}_0(T_{m-a})$ computed in 2.

4. Give the linear mapping φ_m by means of a matrix using that

$$\varphi_m(\{e_1\}-\{e_2\}) = \begin{cases} \{e_1\}-\{e_2\} & \text{if } m-e_1-e_2 \notin S \\ 0 & \text{otherwise} \end{cases}$$

(see Proposition 3.2 in [2])

Now, let

$$t_1 := \min\{t \mid coker(\varphi_m) = 0 \ \ \forall m \in H^t\},$$

i.e. the minimum $t \in \mathbf{N}$ such that φ_m is surjective for every $m \in H^t$. (There exists t_1 by 3.2)

REMARK 4.3 In order to find t_1 it is enough to check the condition

$$coker(\varphi_m) = 0$$

over the set $D(0)$ computed by 3.3 and using the matrix given in 4.2.

LEMMA 4.4 Let $m \in H^t$ with $t > t_0$, then $\widetilde{H}_{-1}(T_{m-a}) = 0$, for any $a \in A$.

Proof. One has $\widetilde{H}_{-1}(T_{m-a}) \neq 0$ if and only if $T_{m-a} = \{\emptyset\}$, i.e, if and only if $m-a \in S$ and $m-a-e \notin S$, for every $e \in E$. But $m-a-e \notin S$, for all $e \in E$ implies $m-a \in Q$. Since $m-a \in H^{t-1}$, we have that $m-a \in Q^{t-1}, t-1 \geq t_0$. But $Q^{t-1} = \emptyset$ by 2.2. Therefore, $\widetilde{H}_{-1}(T_{m-a}) = 0$. □

THEOREM 4.5 Let S be a semigroup with the previous conditions. Then, the degrees of the polynomials in a minimal generating set of the ideal of S are less or equal than $\max(t_0, t_1)$.

Proof. The degrees we are looking for are

$$\{L(m) \mid \widetilde{h}_0(\Delta_m) \neq 0\}.$$

Hence, it is enough to prove that if $t > \max(t_0, t_1)$, then $\widetilde{H}_0(\Delta_m) = 0$ for any $m \in H^t$.

First, since $t > t_0$ we obtain the exact sequences:

$$\ldots \to H_1(\mathbf{Q}_m) \to \widetilde{H}_1(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_0(\Delta_m) \to 0 = H_0(\mathbf{Q}_m),$$

$$.. \to \bigoplus_{m \in A} \widetilde{H}_0(T_{m-a}) \stackrel{\varphi_m}{\to} \widetilde{H}_0(T_m) \to \widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(-1)}) \to 0.$$

Second, since $t > t_1$, we obtain $\widetilde{H}_1(\mathbf{M}_m^{(1)}, \mathbf{M}_m^{(-1)}) = 0$. Now, the diagram guarantees that $\widetilde{H}_1(\overline{\mathbf{K}}_m, \mathbf{K}_m) = 0$, and the result in the theorem follows from this fact.

□

5 AN EXAMPLE: THE DEGREE OF THE EQUATIONS OF TORIC PROJECTIVE CURVES

A projective toric curve is the projective scheme $Proj(k[S])$, where k is the ground field and S is the subsemigroup of $\mathbf{N}^2$ generated by elements of a set Λ consisting of elements of type

$$e_1 = (d, 0), e_2 = (0, d), a_1 = (a_{11}, a_{12}), ..., a_r = (a_{r1}, a_{r2}),$$

where $d > 0$ is the degree of the curve, $a_{i1} + a_{i2} = d$ for each i, and $gcd(d, a_{11}, ..., a_{1r}) = 1$. The choice of the generators gives an embedding of the toric curve into the projective space $\mathbf{P}^{r+1}$. The homogeneous ideal defining this embedding is nothing but the ideal associated to the semigroup S and the generator set Λ. Notice that the pair S,Λ satisfies the conditions in section 1, if one chooses the map L given by $L(a_1, a_2) = (a_1 + a_2)/d$. Hence the degrees of the homogeneous equations defining the toric projective curve can be estimated from the results in above sections.

For it, consider the partition $\Lambda = E \cup A$ where $E = \{e_1, e_2\}$ and $A = \{a_1, ..., a_r\}$. Then, for each $m \in S$, the simplicial complex T_m has non trivial reduced homology if either it consists of only the empty set (in that case the -1 reduced homology is isomorphic to k) or it is the complex T consisting of the two points corresponding to e_1 and e_2 but not to the edge joining both points (in that case the 0-th homology is isomorphic to k). Thus, the estimation of the degrees of the equations will be given in terms of the sets Q and D where Q is the Apery set (which can be viewed as the set of $m \in S$ such that $T_m = \{\emptyset\}$) and D is the set of those $m \in S$ such that $T_m = T$. Note that the set D is empty if and only if the curve is arithmetically Cohen-Macaulay, i.e. if the k-algebra $k[S]$ is Cohen-Macaulay (see [8]).

Now, in the Cohen-Macaulay case, the degree of the equations of the curve are bounded by the integer t_0 defined in section 1. In the non Cohen Macaulay case, one can also define the integer t_2 to be the largest integer t such that $D \cap H^t$ is non empty. Notice that one has $t_1 \leq t_2$, so the equations of the projective curve are bounded by the integer $max(t_0, t_2)$.

Using the explicit description, given in [2], of the sets Q and D in terms of the numerical semigroup S_1 generated by the integers $d, a_{11}, ..., a_{1r}$ 4.5, the integers t_0 and t_2 can be easily computed as follows. First, for every $b \in \mathbf{Z}$, set $l(b)$ equal to infinity if $b \notin S$ and, otherwise, equal to the least number of integers among $d, a_{11}, ..., a_{1r}$ (allowing repetitions) with sum equal to b. Then, if B is the set of $b \in S$ such that $l(b - d) \leq l(b)$, and if B' is the subset of B consisting of those $b \in B$ such that $b - d \in S$, one has

$$t_0 = 1 + max\{l(b) \quad | \quad b \in B\},$$

$$t_2 = max\{l(b - d) \quad | \quad b \in B'\}.$$

Thus, one concludes the following result:

THEOREM 5.1 With assumptions and notations as above, the degrees of the polynomials in a minimal set of homogeneous equations defining a toric projective curve C are bounded by the integer

$$1 + max\{l(b) \quad | \quad b \in B\}$$

if C is arithmetically Cohen Macaulay, and by

$$min\{1 + max\{l(b) \quad | \quad b \in B\}, max\{l(b - d) \quad | \quad b \in B'\}\},$$

if C is not arithmetically Cohen-Macaulay.

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