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Simplicial Complexes and Syzygies of Lattice Ideals

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Simplicial Complexes and Syzygies of Lattice Ideals*

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Abstract

We compute the degrees of the syzygies of lattice ideals by means of some simplicial complexes. The vertex sets of those complexes are subsets of the extremal rays of the associated cone. In particular, we deduce a way of computing the depth of the ideal. In the case when the lattice ideal is homogeneous, the regularity is computed in terms of such complexes.

Introduction

Toric ideals with the terminology in [12] are special cases lattice ideals. Namely, lattice ideals can be seen as a generalization of toric ideals when one allows torsion in the semigroup S which parametrizes the variety.

A determination of the degrees of the syzygies of these ideals by means of some simplicial complexes can be found in [8]. These simplicial complexes have their vertices on a generating set of S, Λ.

In this paper, we adapt and extend the methods in [8] and give a method of computing such degrees by means of other simplicial complexes appeared in [9]. The advantage of these new complexes is that they are constructed taking their vertices on a subset of semigroup generators denoted by E, which in practice is much more small. Precisely, a generator is taken over each extremal ray of the associated cone.

The method uses the Apery set Q, relative to the subset of the generator chosen, in addition to Hilbert bases of some diophantine systems. Both sets are finite and can be computed by Integer Programming methods. From the characterization of the depth in [9] we deduce how such characterization (in terms of above simplicial complexes) becames, now, an effective one.

The results in this paper are significant when $E \neq \Lambda$. However, we remark that there some interesting cases satisfying $E = \Lambda$, as, for instance, the Lawrence ideals (see [3], [12] and [1]). In these cases $Q = \{0\}$, and all our results coincide with the results in [8].

The last part of the paper considers the projective case. First Proposition 17 gives a characterization of when a lattice ideal is homogeneous which generalizes

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Lemma 4.14 in [12]. Finally, we give a characterization of the regularity of a lattice ideal in terms of the considered complexes with vertices in E (see [7] too).

1 i-Triangulations in a simplicial complex

Let Δ be an abstract simplicial complex with vertices over a finite set Λ. This means that $\Delta \subset \mathcal{P}(\Lambda)$, that $\emptyset \in \Delta$ and that if $F \in \Delta$, then $F' \in \Delta$, for any $F' \subset F$.

If $F \in \Delta$ and $\sharp F = d+1$, F is said to be a face of Δ of dimension d, $\dim F = d$. In particular, $\dim\{\emptyset\} = -1$.

Fix an orientation on each face of Δ, and consider the augmented chain complex with values in a field k, i.e., consider the following objects:

- $\tilde{C}_i(\Delta)$ the $k-$vector space generated freely by the i-dimensional faces of Δ;
- $\delta_i : \tilde{C}_i(\Delta) \to \tilde{C}_{i-1}(\Delta)$ the $k-$linear mapping given by

$$\delta_i(F) = \sum_{F' \in \Delta, \dim F' = i-1} \epsilon_{FF'} F',$$

  where $\epsilon_{FF'} = 0$ if $F' \not\subset F$, and $\epsilon_{FF'} = \pm 1$ if $F' \subset F$, $\epsilon_{FF'} = 1$ if the orientation induced by F on F' is equal to the orientation chosen on F', and $\epsilon_{FF'} = -1$ otherwise.
- $\tilde{Z}_i(\Delta) = ker(\delta_i)$ and $\tilde{B}_i(\Delta) = Im(\delta_{i+1})$ are the spaces of cycles and boundaries respectively.

Hence, the reduced i-homology of the simplicial complex Δ is the k-vector space

$$\tilde{H}_i(\Delta) = \tilde{Z}_i(\Delta)/\tilde{B}_i(\Delta).$$

Definition 1. *Let $i \geq 0$ and $F \subset \Lambda$. We will say that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F if the following properties are satisfied:*

1. $\sharp F_j = i+1$, $\forall j = 1, \ldots, t$.
2. $F = \bigcup_{j=1}^{t} F_j$.

We will say that τ is an i-triangulation of F in Δ, if $F_j \in \Delta$, $\forall j = 1, \ldots, t$, and $F \notin \Delta$.

The following on nonvanishing of the homology will be used later in this paper.

Lemma 2. *Assume that $\tilde{H}_i(\Delta) \neq 0$, and let $c \in \tilde{Z}_i(\Delta) - \tilde{B}_i(\Delta)$, $c = \sum_{j=1}^{t} \lambda_j F_j$, $\lambda_j \in k - \{0\}$ for any $j = 1, \ldots, t$, $F_j \neq F_l$ if $j \neq l$. Then, if $F = \bigcup_{j=1}^{t} F_j$ one has that*

$$\forall p \in F \quad \exists j, 1 \leq j \leq t \quad | \quad F_j \cup \{p\} \notin \Delta.$$

Proof. See [8],1.1. □

Corollary 3. *If $\tilde{H}_i(\Delta) \neq 0$, there exists $c \in \tilde{Z}_i(\Delta) - \tilde{B}_i(\Delta)$, $c = \sum_{j=1}^{t} \lambda_j F_j$, such that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F in Δ, for $F = \bigcup_{j=1}^{t} F_j$.*

Proof. It is enough to take $c \in \tilde{Z}_i(\Delta) - \tilde{B}_i(\Delta)$ and $F, F_1, \ldots, F_t$ as in Lemma 2. □

2 Simplicial Complexes associated to a Semigroup

Fix k a commutative field, S a cancellative finitely generated commutative semigroup satisfying the property $S \cap (-S) = (0)$, and Λ a system of generators of S, $\sharp\Lambda = h$. Let $G(S)$ be the smallest group containing S, and $V(S) := G(S) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let $C(S)$ be the cone generated by S, i.e., the rational cone in $V(S)$ generated by the image $\overline{S}$ of S in $V(S)$. Notice that $C(S)$ is strongly convex because $S \cap (-S) = (0)$.

Let d be the dimension of S, i.e. $d = \text{rk } G(S)$, and let f be the number of extremal rays of the strongly convex cone $C(S)$. Notice that, since $C(S)$ generates $V(S)$, one obviously has $f \geq d$. Then, there are subsets $E \subset \Lambda$ with $\sharp E = f$ such that $C(E) = C(S)$, where $C(E)$ is the cone in $V(S)$ generated by E.

Following [9], fix a partition $\Lambda = E \cup A$, where E satisfies the above property. This kind of partition is called *convex partition*. Let $\sharp A = r = h - f$.

For any $m \in S$, consider the simplicial complexes:

$$\Delta_m = \{F \subset \Lambda \quad | \quad m - n_F \in S\}$$

and

$$T_m = \{F \subset E \quad | \quad m - n_F \in S\},$$

where $n_F := \sum_{n \in F} n$ and $n_\emptyset = 0$.

Our objective will be to give an effective method for computing the finite sets

$$D(i) := \{m \in S \ \ | \ \ \widetilde{H}_i(T_m) \neq 0\},$$

for any $i \geq 0$. The finiteness of $D(i)$ is shown in Proposition 4.1 of [9], a such method is given for $i = -1, 0$ in [6]. For $i = -1$, notice that the set $D(-1)$ is the Apery set Q of S relative to E given by

$$Q := \{m \in S \quad | \quad m - e \notin S, \forall e \in E\}.$$

Since $C(E) = C(S)$ for any element $a \in A$ there exists $q_a \in \mathbf{N}$ such that

$$q_a \cdot a = \sum_{e \in E} \lambda_e \cdot e$$

with $\lambda_e \in \mathbf{N}$. Therefore, Q can be obtained, for example, cheking whether the elements $m = \sum_{a\in A} \lambda_a \cdot a$, with $\lambda_a < q_a$, are in Q.

Our method follows from Proposition 11, needs the set Q, and it works for any $i \geq 0$.

As application of this method, in section 5 the degrees of the syzygies of some ideals will be obtained. To do this, we shall use the following result of [9].

Consider, for any $t \geq 0$, the sets

$$C_t = \{m \in S \mid m = \overline{m}+n_F, \text{ with } \overline{m} \in D(i) \text{ and } F \subset A, \sharp F = t-i, \text{ for some } i \geq -1\}.$$

Proposition 4. *For any $t \geq 0$, one has $\widetilde{H}_t(\Delta_m) = 0$ if $m \notin C_t$.*

Proof. It is the Proposition 3.3 in [9]. □

The above proof and our Theorem 19 need new combinatorical objects:

- For any $m \in G(S)$ and $l \geq -1$, denote by $C_l(\mathbf{Q}_m)$ the vector space which has the set
$$\{L \subset A \quad | \quad \sharp L = l+1, m - n_L \in Q\}$$
as a basis.
- For any chain z in $C_t(\mathbf{Q}_m)$, denote by $\theta_t(z)$ the projection on $C_{t-1}(\mathbf{Q}_m)$ of the simplicial boundary of z.

$\{C_\bullet(\mathbf{Q}_m), \theta_\bullet\}$ is a chain complex for any m (see [9]). To understand better the homology of this complex, consider, for any $m \in S$, the following set:

$$\mathbf{K}_m = \{L \in \Delta_m \quad | \quad (L \cap E \neq \emptyset) \;\; or \;\; (L \subset A \quad \text{and} \quad m - n_L \in S - Q)\}.$$

It is easy to check that $\mathbf{K}_m$ is a simplicial subcomplex of Δ_m, so that one can consider the chain complex $\widetilde{C}_\bullet(\mathbf{K}_m)$ and the relative chain complex $\widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$.

Notice that, by construction, one has an identification $C_\bullet(\mathbf{Q}_m) \simeq \widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$. Moreover if $m \in Q$, then one has that $C_\bullet(\mathbf{Q}_m) \simeq k$ and $\mathbf{K}_m = \emptyset$. Otherwise, if $m \in S - Q$, since $\exists e \in E$ such that $m - e \in S$, one obtains $L = \{e\} \in \mathbf{K}_m$ and $\mathbf{K}_m \neq \{\emptyset\}$.

Therefore, $\widetilde{H}_{-1}(\mathbf{K}_m) = 0$ for any $m \in S$. This allows to deduce, from the exact sequence of complexes,

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\Delta_m) \to C_\bullet(\mathbf{Q}_m) \to 0,$$

that there is a long exact sequence of homology,

$$... \to H_{l+1}(\mathbf{Q}_m) \to \widetilde{H}_l(\mathbf{K}_m) \to \widetilde{H}_l(\Delta_m) \to H_l(\mathbf{Q}_m) \to ...$$

$$... \to \widetilde{H}_0(\mathbf{K}_m) \to \widetilde{H}_0(\Delta_m) \to H_0(\mathbf{Q}_m) \to \widetilde{H}_{-1}(\mathbf{K}_m) = 0 \to$$
$$\widetilde{H}_{-1}(\Delta_m) \to H_{-1}(\mathbf{Q}_m) \to 0.$$

Now, in order to understand the homology $\widetilde{H}_\bullet(\mathbf{K}_m)$, let us consider the simplicial complex given by the following disjoint union of subsets of Σ:

$$\overline{\mathbf{K}}_m := \mathbf{K}_m \cup \{F \cup J, F \subset A, J \subset E \quad | \quad m - n_F - n_J \notin S \text{ and } m - n_F - e \in S, \forall e \in J\}.$$

Notice that any $F \cup J$ in the second set of the above union is such that the cardinality of J is at least 2.

The complex $\overline{\mathbf{K}}_m$ is acyclic, i.e. $\widetilde{H}_l(\overline{\mathbf{K}}_m) = 0$ for any $l \geq -1$ (see Corollary 2.1 in [9]).

Thus, the long exact sequence of homology coming from the exact sequence of chain complexes

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to 0$$

gives rise to an isomorphism $\rho_{l+1} : \widetilde{H}_{l+1}(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_l(\mathbf{K}_m)$, for every $l \geq -1$.

Lemma 5. *With the previous notations:*

a) $\widetilde{H}_l(\Delta_m) \cong \widetilde{H}_{l+1}(\overline{\mathbf{K}}_m, \mathbf{K}_m)$, $\forall l \geq r$.

b) If $\widetilde{H}_r(\overline{\mathbf{K}}_m, \mathbf{K}_m) \neq 0$ *then* $\widetilde{H}_{r-1}(\Delta_m) \neq 0$.

Proof. It is enough to use that $H_l(\mathbf{Q}_m) = 0$, $\forall l \geq r$ and the above sequences. $\square$

To study the homology $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ let us consider, the chain of simplicial complexes

$$\mathbf{K}_m = \mathbf{M}_m^{(-1)} \subset \mathbf{M}_m^{(0)} \subset \mathbf{M}_m^{(1)} \subset ... \subset \mathbf{M}_m^{(r)} = \overline{\mathbf{K}}_m$$

where $\mathbf{M}_m^{(i)}$, $-1 \leq i \leq r$, is the simplicial subcomplex of $\overline{\mathbf{K}}_m$ given by:

$$\mathbf{M}_m^{(i)} := \mathbf{K}_m \cup \{L = F \cup J \in \overline{\mathbf{K}}_m \quad | \quad F \subset A, J \subset E, \quad \text{and} \quad \sharp F \leq i\}.$$

Now, $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ can be computed (see [9]) by means of the long exact sequences

$$... \to \widetilde{H}_l(\mathbf{M}_m^{(j)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(j)}) \to ....$$

for $-1 \leq i < j < k \leq r$. In fact, to compute $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) = \widetilde{H}_\bullet(\mathbf{M}_m^{(r)}, \mathbf{M}_m^{(-1)})$, it will be enough to use the above exact sequences for the concrete values of (i, j, k) given by $(-1, 0, 1), (-1, 1, 2), ..., (-1, r-1, r)$, and take into account the following result which is obvious by construction (see Proposition 3.2 in [9]).

Proposition 6. *With the previous notations, for any* $m \in S$, *one has: for any* $l \geq -1$ *and any* i, $0 \leq i \leq r$,

$$\widetilde{H}_{l+1}(\mathbf{M}_m^{(i)}, \mathbf{M}_m^{(i-1)}) \simeq \bigoplus_{F \subset A, \sharp F = i} \widetilde{H}_{l-i}(T_{m-n_F})$$

(in this formula, $\widetilde{H}_{l-i}(T_{m-n_F}) = 0$ *if either* $l + 1 < i$ *or* $m - n_F \notin S$*).*

3 Finding the finite set D(i)

Fix the same notation than section 2. Notice that since S is finitely generated commutative, we can consider excepting isomorphism

$$G(S) = \mathbf{Z}^d \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/a_s\mathbf{Z},$$

where $a_i \in \mathbf{Z}, 1 \leq i \leq s$. Set $\Lambda = \{n_1, \ldots, n_h\}$ a system of generators of S. Let $\mathcal{A}$ be the matrix whose column vectors are the generators of S, $\mathcal{A} := (n_1|\ldots|n_h) \in \mathcal{M}_{(d+s)\times h}(\mathbf{Z})$, considering the elements n_i as elements in $\mathbf{Z}^{d+s}$, and let

$$\mathcal{A}(t) := \begin{pmatrix} \mathcal{A} & -\mathcal{A} & 0 & 0 & 0 & & 0 & 0 \\ 0 & \mathcal{A} & -\mathcal{A} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{A} & -\mathcal{A} & 0 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \mathcal{A} & -\mathcal{A} \end{pmatrix} \in \mathcal{M}_{(d+s)(t-1)\times ht}(\mathbf{Z}).$$

Let $F \subset E$ and let $\tau = \{F_1, \ldots, F_t\}$ be an i-triangulation of F. Set $e_{F_l} \in \mathbf{N}^h$ the vector with coordinates equal to zero, excepting the jth one which is equal to one, for any $j \in F_l$. Set $e_\tau := (e_{F_1}, \ldots, e_{F_t}) \in \mathbf{N}^{ht}$, and set

$$R_\tau := \{\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in \mathbf{N}^{ht} \mid \mathcal{A}(t)\alpha = 0 \;\;, \alpha \gg e_\tau\},$$

where $\gg$ stands for the componentwise partial order in $\mathbf{N}^{ht}$.

Notice that if $\alpha \in R_\tau$ and it is written $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)})$ with $\alpha^{(j)} \in \mathbf{N}^h$, then for any $j, 1 \leq j \leq t$, one obtains $\mathcal{A}\alpha^{(1)} = \cdots = \mathcal{A}\alpha^{(t)} = m \in S$ for some m.

Set

$$\Sigma R_\tau := \{m \in S \mid m = \mathcal{A}\alpha^{(1)}, \text{ for some } \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in R_\tau\}.$$

Lemma 7. *Let $m \in S$, $F \subset E$ and let $\tau = \{F_1, \ldots, F_t\}$ be an i-triangulation of F in T_m, then $m \in \Sigma R_\tau$.*

Proof. It is enough to use that $F_j \in T_m$ if and only if there exists $\alpha^{(j)} \in \mathbf{N}^h$ such that $\mathcal{A}\alpha^{(j)} = m$, and $\alpha^{(j)} \gg e_{F_j}$.

□

It is well-known that the set $\mathcal{H}R_\tau := \{\alpha \in R_\tau \mid \alpha \text{ is minimal for } \ll\}$ is finite, hence the set

$$\Sigma\mathcal{H}R_\tau := \{m \in S \mid m = \mathcal{A}\alpha^{(1)}, \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in \mathcal{H}R_\tau\},$$

is finite.

Definition 8. *On the elements of S, define a partial order $>_Q$*

$$m >_Q m' \iff m - m' \in S - Q.$$

If $H \subset S$, we say that $m \in H$ is Q-minimal in the set H if $m \geq_Q m'$, with $m' \in H$, implies that $m = m'$.

Set

$$M_\tau := \{m \in \Sigma R_\tau \ \mid \ m \text{ is } Q\text{−minimal in } \Sigma R_\tau\}.$$

Lemma 9. *In the conditions as above, for any $m \in \Sigma R_\tau$, there exists $m' \in M_\tau$ such that $m = m' + m''$, with $m'' = 0$ or $m'' \in S - Q$.*

Proof. It is a consequence of that the condition $S \cap (-S) = \{0\}$ guarantees that the number of different expressions of m as sum of non null elements in S is finite (Proposition 1.2 in [4]).

□

Lemma 10. *In the conditions as above, $M_\tau \subset \Sigma\mathcal{H}R_\tau + Q$. Therefore, the set M_τ is finite.*

Proof. Let $m \in M_\tau$. Then, $m = \mathcal{A}\alpha^{(1)}$ with $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in R_\tau$. If $\alpha \in \mathcal{H}R_\tau$, we are finished. Suppose that $\alpha \notin \mathcal{H}R_\tau$. Then $\alpha = \alpha' + \alpha''$, with $\alpha' \in \mathcal{H}R_\tau$ and $\alpha'' \in \mathbf{N}^{ht}$. Then, if $m' = \mathcal{A}\alpha'^{(1)}, m' \in \Sigma\mathcal{H}R_\tau$. By $m - m' = m'' = \mathcal{A}\alpha''^{(1)} \in S$ and m is Q−minimal, if $m'' \neq 0$ we obtain that $m'' \in Q$.

□

Proposition 11. *The set*

$$D(i) := \{m \in S \ \mid \ \widetilde{H}_i(T_m) \neq 0\}$$

is finite and it can be determined by an algorithm.

Proof. Consider $m \in D(i)$. Notice that it is enough to prove that $m \in M_\tau$ for some i-triangulation τ of F in T_m.

By corollary 3 we obtain that there exists $c \in \tilde{Z}_i(T_m) - \tilde{B}_i(T_m)$, $c = \sum_{j=1}^{t} \lambda_j F_j$, such that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F in T_m, for $F = \bigcup_{j=1}^{t} F_j$.

If $m \in M_\tau$, we are finished. Otherwise, by lemmas 7 and 9, $m = m' + m''$ with $m' \in M_\tau$ and $m'' \in S - Q$. Set $E = \{n_1, \ldots, n_f\}$ and $A = \{n_{f+1}, \ldots, n_{f+r}\}$, we can write $m'' = \sum_{j=1}^{f+r} \beta_j n_j$, $\beta_j \in \mathbf{N}$ and $\beta_j \neq 0$ for some j, $1 \leq j \leq f$.

Suppose that $\beta_1 \neq 0$. If $n_1 \in F$, we apply lemma 2 for $p = 1$. Then, there exists $j, 1 \leq j \leq t$, such that $n_1 \notin F_j$ and $F_j \cup \{n_1\} \notin T_m$. However, $m' - n_{F_j} \in S$ because $m' \in M_\tau$, and $m'' - n_1 \in S$ because $\beta_1 \neq 0$. This means that $m - n_{F_j} - n_1 \in S$ which is a contradiction with $F_j \cup \{n_1\} \notin T_m$. Therefore, $n_1 \notin F$.

By $n_1 \notin F$ we obtain in a similar above way that $m - n_{F_j} - n_1 \in S$, for any j. But then $F'_j = F_j \cup \{n_1\} \in T_m$ and

$$c' = \sum_{j=1}^{t} \lambda_j F'_j \in \tilde{C}_{i+1}(T_m).$$

Then, we have that $\delta_{i+1}(c') = c$, which is a contradiction because $c \notin \tilde{B}_i(T_m)$. Therefore $m'' = 0$, and our result is proved.

□

Remark 12. *The following construction makes effective the characterization of* $q := depth_R\ k[S]$ *given in Theorem 4.1 in [9]. Moreover, it is an effective method to check the Cohen-Macaulay property* ($d = q$).

The set $D(i)$ *can be computed with the following steps:*

1. *Compute the set* Q.
2. *Compute* $\Sigma\mathcal{H}R_\tau$, *for any* i*-triangulation* τ *of* F, $F \subset E$, $\sharp F \geq i+2$. *(Use Integer Programming, see for example [11])*
3. *Check the* Q*-minimal elements in the set*

$$\Sigma\mathcal{H}R_\tau + Q$$

to obtain M_τ, *for any* i*-triangulation* τ *of* F, $F \subset E$, $\sharp F \geq i+2$.

4. *Determine the set*

$$G = \{(m, \tau, F) \ \mid \ m \in M_\tau,\ \tau \ \ i-triangulation\ of\ \ F \ \ in \ \ T_m\}.$$

5. *Check whether* $\widetilde{H}_i(T_m) \neq 0$ *on the set*

$$\{m \in S \ \mid \ (m, \tau, F) \in G\}.$$

4 Degrees of syzygies of lattice ideals

We use the notation of precedent sections. Let $k[S]$ be the semigroup k-algebra associated to S, which is considered by $k[S] = \bigoplus_{m\in S} k\chi^m$, where $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. Let $R = k[X_1, \ldots, X_h]$ be the polynomial ring in h variables.

$k[S]$ is obviously an S-graded ring, and R is S-graded assigning the degree n_i to X_i. Let $\mathbf{m}$ be the irrelevant ideal of R. The $k-$algebra morphism,

$$\varphi : R \longrightarrow k[S],$$

defined by $\varphi(X_i) = \chi^{n_i}$, is an $S-$graded morphism of degree zero. Thus, the ideal $I = ker(\varphi)$ is an $S-$homogeneous ideal. Since φ is surjective, $k[S] \simeq R/I$.

I is a lattice ideal because it is the ideal associated to the lattice

$$ker(S) := \{\mathbf{u} \in \mathbf{Z}^h \ \mid \ \sum_{i=1}^{h} n_i u_i = 0\}$$

(see Lemma 9 in [13]). This means that

$$I = \langle \mathbf{X}^{u^+} - \mathbf{X}^{u^-} \ \mid \ \mathbf{u} \in ker(S)\rangle,$$

where $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ with $\mathbf{u}^+, \mathbf{u}^- \in \mathbf{N}^h$, and $\mathrm{supp}(\mathbf{u}^+) \cap \mathrm{supp}(\mathbf{u}^-) = \emptyset$.

Notice that the property $S \cap (-S) = (0)$ is equivalent to the property $ker(S) \cap \mathbf{N}^h = (0)$.

On the other hand, the condition $S \cap (-S) = \{0\}$ guarantees the $S-$graded Nakayama's lemma (Proposition 1.4 in [4]). Then, there exists a minimal $S-$graded free resolution of $k[S]$.

Let N_i be the corresponding i-syzygy module ($N_0 = I$) and consider the $k-$vector spaces

$$V_i(m) := \frac{(N_i)_m}{(\mathbf{m}N_i)_m}, m \in S.$$

In particular, since R is noetherian, one has $V_i(m) = 0$ for all m but finite many of values. Set

$$S(i) := \{m \in S \ \mid \ V_i(m) \neq 0\},$$

the set of S-degrees for the minimal i-syzygies of $k[S]$.

Proposition 13. *The set of S-degrees for the minimal i-syzygies of $k[S]$ can be computed from the simplicial complexes T_m, $m \in S$.*

Proof. We are looking for the set $S(i)$. There exists an effective isomorphism

$$\tilde{H}_i(\Delta_m) \simeq V_i(m),$$

for any $m \in S$ (see [5]). Then,

$$S(i) = \{m \in S \ \mid \ \tilde{H}_i(\Delta_m) \neq 0\}.$$

Proposition 4 guarantees that $S(i) \subset C_i$, hence it is enough to compute the set C_i from the simplicial complexes T_m, $m \in S$. But it is possible by using Proposition 11.

□

Remark 14. *To find the S-degrees for the minimal i-syzygies of $k[S]$ one can do the following:*

1. *Compute $D(j)$, for any $j \leq i$. (See Remark 12)*
2. *Compute the set C_i. (It is enough to use the definition)*
3. *Check the elements $m \in C_i$ such that $\tilde{H}_i(\Delta_m) \neq 0$ and obtain $S(i)$.*

Theorem 15. *The minimal S-graded free resolution of $k[S]$ can be computed from the simplicial complexes T_m, $m \in S$.*

Proof. It is enough to take the images of the elements in a basis for the i-reduced homology space $\tilde{H}_i(\Delta_m)$ by the effective isomorphism

$$\tilde{H}_i(\Delta_m) \simeq V_i(m),$$

for any $m \in S(i)$.

□

Remark 16. *The minimal S-graded free resolution of $k[S]$ can be computed doing:*

1. *For any i, $1 \leq i \leq h-2$:*
   - *Compute $S(i)$. (Remark 14)*
   - *Compute a basis for the i-reduced homology $\tilde{H}_i(\Delta_m)$. (Using linear Algebra and integer Programming)*
   - *Take the images of the element in the above basis by the effective isomorphism*
   $$\tilde{H}_i(\Delta_m) \simeq V_i(m)$$
   (Remark 3.6 in [5]), and obtain a minimal generating set of the i-syzygy module of $k[S]$, N_i.
2. *The minimal S-graded free resolution of $k[S]$ is:*
$$0 \to R^{b_{h-1}} \overset{\varphi_{h-1}}{\to} \cdots \to R^{b_2} \overset{\varphi_2}{\to} R^{b_1} \overset{\varphi_1}{\to} R \overset{\varphi_0}{\to} k[S] \to 0,$$
where for any i, $1 \leq i \leq h-1$, $b_i = \sum_{m \in S(i-1)} dim_k \tilde{H}_{i-1}(\Delta_m)$ and the mapping φ_i is given by the matrix whose column vectors are generators of N_{i-1} $(N_0 = I)$.

Notice that if

$$p := \max\{i \mid b_i \neq 0, \ \ 1 \leq i \leq h-1\},$$

the Auslander-Buchbaum theorem guarantees that $p = h - \mathrm{depth}_R\, k[S]$.

5 The regularity of a homogeneous lattice ideal

Let $\mathcal{L} \subset \mathbf{Z}^h$ be a lattice, and let I be the ideal associated to $\mathcal{L}$, i.e.

$$I = \langle \mathbf{X}^{u^+} - \mathbf{X}^{u^-} \mid \mathbf{u} \in \mathcal{L} \rangle.$$

We can consider the semigroup $S \subset \mathbf{Z}^h/\mathcal{L}$ generated by $\{e_1 + \mathcal{L}, \ldots, e_h + \mathcal{L}\}$, where the e_i are the unit vectors in $\mathbf{N}^h$. Set $n_i = e_i + \mathcal{L}$, $1 \leq i \leq h$. Then, $I = \ker(\varphi)$, where as in the precedent section,

$$\varphi : R \longrightarrow k[S]$$

is defined by $\varphi(X_i) = \chi^{n_i}$.

The following result characterizes when the ideal I is homogeneous for the natural grading. We will not use the special form of the generators of S. This is not strange because the lattice ideals and the ideals of finitely generated commutative semigroups are the same thing (Lemma 9 in [13]). Here, it is not necessary the condition $S \cap (-S) = (0)$.

We can suppose that

$$S \subset \mathbf{Z}^d \oplus \mathbf{Z}/a_1\mathbf{Z} \oplus \cdots \oplus \mathbf{Z}/a_s\mathbf{Z},$$

with $a_i \in \mathbf{Z}$ non null, $1 \leq i \leq s$.

Let π be the projection over the first coordinates

$$\pi : \mathbf{Z}^{d+s} \longrightarrow \mathbf{Z}^d.$$

Proposition 17. *With the above notations, I is homogeneous for the natural grading if and only if there exists $\mathbf{w} \in \mathbf{Q}^d$ such that $\mathbf{w} \cdot \pi(n_i) = 1$, for any $i = 1, \ldots, h$.*

Proof. Let S_1 be the subsemigroup of $\mathbf{Z}^{d+s}$ generated by $\{n'_1, \ldots, n'_{h+s}\}$, where

- if $1 \leq i \leq h$, $n'_i = n_i$ considering n_i as an element of $\mathbf{Z}^{d+s}$,
- if $h+1 \leq i \leq h+s$, n'_i is the vector in $\mathbf{Z}^{d+s}$ with coordinates equal to zero, excepting the $(d+i-h)$-th one which is equal to a_{i-h}.

Let $\pi' : \mathbf{Z}^{h+s} \longrightarrow \mathbf{Z}^h$ be the projection over the first coordinates.

Notice that $\mathcal{L}_1 := \pi'(ker(S_1))$ is a lattice of $\mathbf{Z}^h$, moreover, $\mathcal{L}_1 = ker(S)$.

On the other hand, consider S_2 the subsemigroup of $\mathbf{Z}^d$ generated by $\pi(n_i)$, $1 \leq i \leq h$. Set $\mathcal{L}_2 := ker(S_2)$ which is another lattice of $\mathbf{Z}^h$.

It is easy to see that $\mathcal{L}_1 \subset \mathcal{L}_2$. We claim that both lattices have the same rank. In fact, it is clear that $\mathrm{rk}(\mathcal{L}_1) \leq \mathrm{rk}(ker(S_1)) = (d+s) - \mathrm{rk}(\mathcal{A}')$, where $\mathcal{A}' := (n'_1|\ldots|n'_{h+s}) \in \mathcal{M}_{(d+s)\times(h+s)}(\mathbf{Z})$. From the special form of this matrix follows that if $\mathcal{C}$ is a basis of $ker(S_1)$, then $\pi'(\mathcal{C})$ is a basis of $\pi'(ker(S_1))$. Therefore, $\mathrm{rk}(\mathcal{L}_1) = \mathrm{rk}(ker(S_1))$. Notice also that

$$\mathrm{rk}(\mathcal{A}') = s + \mathrm{rk}(\pi(n_1)|\ldots|\pi(n_h)),$$

which implies that

$$\mathrm{rk}(ker(S_1)) = d - \mathrm{rk}(\pi(n_1)|\ldots|\pi(n_h)) = \mathrm{rk}(\mathcal{L}_2).$$

Therefore, we have prove that

$$\mathrm{rk}(\mathcal{L}_1) = \mathrm{rk}(\mathcal{L}_2).$$

On the other hand, notice that it is satisfied that

$$I = \langle \mathbf{X}^{u^+} - \mathbf{X}^{u^-} \ \mid \ \mathbf{u} \in \mathcal{L}_1 \rangle.$$

Therefore, I is homogeneous if and only if $(1, \ldots, 1) \in \mathcal{L}_1^{\perp}$. Since $\mathrm{rk}(\mathcal{L}_1) = \mathrm{rk}(\mathcal{L}_2)$ and $\mathcal{L}_1 \subset \mathcal{L}_2$, it holds if and only if $(1, \ldots, 1) \in \mathcal{L}_2^{\perp}$, and hence if and only if there exists $\mathbf{w} \in \mathbf{Q}^d$ such that $\mathbf{w} \cdot \pi(n_i) = 1$, for any $i = 1, \ldots, h$.

□

Lemma 18. *With the above notations, if I is homogeneous, then $S \cap (-S) \neq (0)$.*

Proof. Suppose that $S \cap (-S) \neq (0)$, then there exist $\alpha_i \in \mathbf{N}$ such that $\sum_{i=1}^h \alpha_i n_i = 0$, with some $\alpha_i \neq 0$. Hence, $\sum_{i=1}^h \alpha_i \pi(n_i) = 0$. Let $\mathbf{w} \in \mathbf{Q}^d$ be the vector given by Proposition 17. We obtain $\sum_{i=1}^h \alpha_i(\pi(n_i) \cdot \mathbf{w}) = \sum_{i=1}^h \alpha_i = 0$, that is a contradiction.

□

The following theorem is a formula (recently obtained in [7]) for the regularity of I in terms of the simplicial complexes T_m. Above computations makes effective such formula. For the sake of completeness of the paper, we prove the result.

Notice that if I is homogeneous and $m \in S$, it is well defined $||m|| = ||\alpha||_1$, where $m = \mathcal{A}\alpha$ and $||\alpha||_1 = \sum_{i=1}^{h} \alpha_i$.

Theorem 19. *With the above notations, assume that I is homogeneus, then*

$$reg(I) = max_{-1 \leq i \leq f-2}\{u_i - i\},$$

where $u_i = max\{||m|| \mid m \in D(i)\}$.

Proof. The regularity of I is $\text{reg}(I) = max_{0 \leq i \leq h-2}\{t_i - i\}$, where t_i is the maximum degree of the minimal i-syzygies of I (see, for example, [2]).

Let i be such that $0 \leq i \leq h-2$ and $m \in S(i)$ such that $t_i = ||m||$. Since $S(i) \subset C_i$ (Proposition 4), $m = \overline{m} + n_F$ where $\overline{m} \in D(j)$ for some $-1 \leq j \leq f-2$, $F \subset A$ and $\sharp F = i - j$. Therefore,

$$t_i - i = ||m|| - i = ||\overline{m}|| + (i-j) - i \leq u_j - j.$$

Hence, we obtained that

$$reg(I) \leq max_{-1 \leq i \leq f-2}\{u_i - i\}.$$

In order to prove the contrary inequality, we consider $M := max_{-1 \leq i \leq f-2}\{u_i - i\}$, and the finite set $P := \{(m,i) \mid m \in S, ||m|| = u_i, \text{ and } M = ||m|| - i\}$.

Take $(m,i) \in P$ with maximum $||m||$. Let $\overline{m} := m + n_A$ and $t := i + r$. We claim that $\widetilde{H}_t(\Delta_{\overline{m}}) \neq 0$.

Our claim proves the contrary inequality because $\overline{m} \in S(t)$ and therefore, $M = ||m|| - i = ||\overline{m}|| - t \leq reg(I)$.

In order to prove the claim: let $F \subset A$ with $\sharp F = j$, where $0 \leq j \leq r-1$. If $\widetilde{H}_{t-j}(T_{\overline{m}-n_F}) \neq 0$, since $||\overline{m} - n_F|| - (t-j) = ||m|| - i$ we obtain that $(\overline{m} - n_F, t-j) \in P$. But this is a contradiction because $||\overline{m} - n_F|| > ||m||$. Therefore $\widetilde{H}_{t-j}(T_{\overline{m}-n_F}) = 0$. A similar reasoning provides that $\widetilde{H}_{t-j-1}(T_{\overline{m}-n_F}) = 0$.

Using the Proposition 6:

a) $\widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(r)}, \mathbf{M}_{\overline{m}}^{(r-1)}) \cong \widetilde{H}_{t-r}(T_m)$.

b) $\widetilde{H}_l(\mathbf{M}_{\overline{m}}^{(j)}, \mathbf{M}_{\overline{m}}^{(j-1)}) = 0$, for any $j = 0, \ldots, r-1$, and for $l = t, t+1$.

From (b) and the following diagram of exact sequences:

$$\cdots \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(0)}, \mathbf{M}_{\overline{m}}^{(-1)}) = 0 \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(1)}, \mathbf{M}_{\overline{m}}^{(-1)}) \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(1)}, \mathbf{M}_{\overline{m}}^{(0)}) = 0 \to \cdots$$

$$\cdots \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(1)}, \mathbf{M}_{\overline{m}}^{(-1)}) = 0 \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(2)}, \mathbf{M}_{\overline{m}}^{(-1)}) \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(2)}, \mathbf{M}_{\overline{m}}^{(1)}) = 0 \to \cdots$$

$$\vdots$$

$$\cdots \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(r-1)}, \mathbf{M}_{\overline{m}}^{(-1)}) = 0 \to \begin{array}{c} \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(r)}, \mathbf{M}_{\overline{m}}^{(-1)}) \\ \| \\ \widetilde{H}_{t+1}(\overline{\mathbf{K}}_{\overline{m}}, \mathbf{K}_{\overline{m}}) \end{array} \to \widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(r)}, \mathbf{M}_{\overline{m}}^{(r-1)}) \to 0 \cdots$$

is obtained that

$$\widetilde{H}_{t+1}(\mathbf{M}_{\overline{m}}^{(r)}, \mathbf{M}_{\overline{m}}^{(r-1)}) \cong \widetilde{H}_{t+1}(\overline{\mathbf{K}}_{\overline{m}}, \mathbf{K}_{\overline{m}}).$$

Therefore, using a) we obtain

$$(*)\ \widetilde{H}_{t+1}(\overline{\mathbf{K}}_{\overline{m}}, \mathbf{K}_{\overline{m}}) \cong \widetilde{H}_i(T_m) \neq 0.$$

If $i \geq 0$, by $(*)$ and a) of Lemma 5, it is obtained that

$$\widetilde{H}_i(T_m) \cong \widetilde{H}_t(\Delta_{\overline{m}}).$$

If $i = -1$, then by $(*)$ and b) of Lemma 5, it is obtained $\widetilde{H}_{r-1}(\Delta_{\overline{m}}) \neq 0$. Now, the claim is proved.

□

Remark 20. *If I is a homogeneous lattice ideal, the regularity of I can be computed with the following steps:*

1. *Compute $D(i)$, for any i, $-1 \leq i \leq f-2$. (Remark 12)*
2. *Compute*
$$u_i = max\{||m|| \mid m \in D(i)\},$$
for any i, $-1 \leq i \leq f-2$.
3. *$reg(I) = max_{-1 \leq i \leq f-2}\{u_i - i\}$.*

Finally, notice that the computation of the regularity does not require the computation of the complete minimal resolution, neither the computation of $S(i)$, where $0 \leq i \leq h-2$, it is enough to determine the sets $D(i)$, where $-1 \leq i \leq f-2$.

References

[1] BAYER D.; POPESCU S.; STURMFELS B. *Syzygies of unimodular lattice ideals.* Preprint math.AG/9912247, (1999).

[2] D. BAYER, M. STILLMAN, *A criterion for detecting m-regularity.* Inventiones mathematicae, **87** (1987), 1-11.

[3] A. BJÖRNER, M. LAS VERGNAS, B. STURMFELS, N. WHITE and G. ZIEGLER, *Oriented Matroids*, Cambridge University Press, (1993).

[4] E. BRIALES, A. CAMPILLO, C. MARIJUÁN, P. PISÓN, *Minimal Systems of Generators for Ideals of Semigroups*, J. of Pure and Applied Algebra, **124** (1998), 7-30.

[5] E. BRIALES, A. CAMPILLO, C. MARIJUÁN, P. PISÓN, *Combinatorics of syzygies for semigroup algebra.* Collet. Math. **49** (1998), 239-256.

[6] E. BRIALES, A. CAMPILLO, P. PISÓN, *On the equations defining toric projective varieties.* Prepublicaciones del Departamento de Álgebra de la Universidad de Sevilla, **5**, (2000). (*Proceeding of the Conference on Conmutative Algebra and Algebraic Geometry in Messina, Italy, 1999*)(to appear in Lect. Notes in Pure and Applied Mathematics.)

[7] E. BRIALES, A. CAMPILLO, P. PISÓN, A. VIGNERON *The regularity of lattice ideals.* Preprint (2000).

[8] E. BRIALES; P. PISÓN; A. VIGNERON *The Regularity of a Toric Variety*, Preprint of University of Sevilla Sección Álgebra **53** (1999). *(Journal of Algebra, to appear).*

[9] A. CAMPILLO, P. GIMÉNEZ, *Syzygies of affine toric varieties.* Journal of Algebra **225**, (2000), 142-161.

[10] P. PISÓN-CASARES, A. VIGNERON-TENORIO, *First Syzygies of Toric Varieties and Diophantine Equations in Congruence.* Preprint of University of Sevilla Sección Álgebra **52** (1999).(*Communications in Algebra to appear*)

[11] P.PISÓN-CASARES, A. VIGNERON-TENORIO, $\mathbf{N}-$*solutions to linear systems over* $\mathbf{Z}$. Preprint of University of Sevilla **43** (1998).

[12] B. STURMFELS, *Gröbner Bases and Convex Polytopes*, AMS University Lectures Series, Vol. 8 (1995).

[13] A. VIGNERON-TENORIO, *Semigroup Ideals and Linear Diophantine Equations.* Linear Algebra and its Applications, **295** (1999), 133-144.