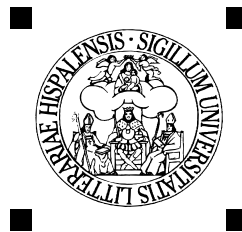


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Computing the rational torsion of an elliptic curve using Tate normal form

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Abstract. It is a classical result (apparently due to Tate) that all elliptic curves with a torsion point of order n ($4 \leq n \leq 10$, or $n = 12$) lie in a one-parameter family. However, this fact does not appear to have been used ever for computing the torsion of an elliptic curve. We present here a extremely down-to-earth algorithm using the existence of such a family.

Mathematics Subject Classification (2000): 11G05

1. Tate and Weierstrass normal forms

An elliptic curve is a plane smooth affine (respectively projective) curve defined by a cubic (homogeneous) polynomial. All these curves are known to be birationally equivalent (that is, isomorphic as algebraic varieties, up to a finite number of points) to one which equation has the form $Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$. When all the coefficients lie in a field K , the set of points in the curve with both coordinates in K admits a group structure ([Cassels 1966], [Cassels 1991], [Husemoller 1987]) with the inner operation defined by the classical chord-tangent procedure. This group is then noted $E(K)$. For historical reasons we will note this operation additively and so

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we will write $2P$ for $P + P$. As the unit element is usually taken to be the only point at infinity (say \mathcal{O}), we can restrict ourselves to affine points.

The Mordell–Weil theorem states that, if K is a number field, $E(K)$ is always a finitely generated abelian group ([Cassels 1991], [Husemoller 1987]). The torsion subset of $E(K)$ is hence a finite subgroup, noted $E_T(K)$. The strongest result concerning $E_T(\mathbf{Q})$ is due to B. Mazur and explicitly states all groups which can appear as torsion subgroups of elliptic curves defined over \mathbf{Q} :

Theorem (Mazur).– ([Mazur 1977], [Mazur 1978]) Let E be an elliptic curve defined over \mathbf{Q} . Then its torsion group $E_T(\mathbf{Q})$ is either isomorphic to C_n (the cyclic subgroup of n elements) for $n = 1, 2, \dots, 10, 12$ or to $C_2 \times C_{2n}$ for $n = 1, 2, 3, 4$. All of these possibilities actually occur.

The aim of this paper is giving an efficient procedure, different from the usual ones, still very lowbrow, for computing the torsion subgroup of an elliptic curve defined over the rationals. First of all we must put the curve into a more manageable form.

For a general elliptic curve it is known (see, for instance, [Cassels 1966], [Cassels 1991], [Husemoller 1987]) that using linear changes of variables, one can take the equation defining the elliptic curve into an easier one of the type $Y^2 = X^3 + AX + B$. This is known as Weierstrass (short) normal form.

A straightforward computation proves that the only linear changes of variables preserving Weierstrass normal form are those given by

$$\begin{cases} X & \longmapsto & u^2 X' \\ Y & \longmapsto & u^3 Y' \end{cases}$$

for some $u \in \mathbf{Q}$. Such a change takes the curve defined by $Y^2 = X^3 + AX + B$ into the one defined by $Y^2 = X^3 + (A/u^4)X + (B/u^6)$. This argument shows that one can always assume A and B to be in \mathbf{Z} . It also implies that the number A^3/B^2 is an invariant of the equivalence class of elliptic curves in Weierstrass form up to linear changes of variables.

Of course, even if two curves $Y^2 = X^3 + AX + B$ and $Y^2 = X^3 + CX + D$ verify $A^3/B^2 = C^3/D^2$ this does not mean they are equal up to some linear change of variables of the previous form. In fact, it is fairly elementary

proving that this happens if and only if the following condition hold: there exists a rational solution u for the system

$$\left\{ u^4 = \frac{A}{C}, \quad u^6 = \frac{B}{D} \right\},$$

with the obvious arrangements for the cases in which any of the coefficients vanishes.

In addition, if the curve is already known to have one rational point of order $n > 3$, one can choose to put the equation of the curve in the form $Y^2 + bXY + cY = X^3 + dX^2$, also using nothing but linear changes of variables. This second formula is called Tate normal form ([Husemoller 1987]).

2. The Lutz – Nagell theorem

Most classical algorithms for computing rational torsion of elliptic curves are based on the following result, achieved independently by Lutz and Nagell ([Nagell 1935], [Lutz 1937]):

Theorem (Lutz – Nagell).– Let E be an elliptic curve defined over \mathbf{Q} , given by a Weierstrass equation $Y^2 = X^3 + AX + B$ with $A, B \in \mathbf{Z}$, and let $P = (\alpha, \beta) \in E_T(\mathbf{Q})$. Then

- (a) Both α and β are in \mathbf{Z} .
- (b) Either $\beta = 0$ or $\beta^2 | (4A^3 + 27B^2)$.

Clearly $\beta = 0$ is equivalent to $2P = O$ and this 2-torsion part is rapidly computable. For computing the remaining points (if there are any) we simply factorize $\Delta = 4A^3 + 27B^2$. This quantity, called the discriminant of E , will be most important in the sequel. For every square divisor, say m^2 of Δ , we compute the integral solutions to $X^3 + AX + (B - m^2)$. If we actually find an integral root, say n , we only have to check whether (n, m) is a torsion point, which only involves computing, at most, $12(n, m)$, following Mazur's Theorem.

Simple as it is, this algorithm is not very efficient, being its major drawback the necessity of factoring Δ . This is the algorithm presented, for instance, in [Cohen 1993] and [Cremona 1992].

3. Good reduction: a first bound

The first step in our algorithm will be a reasonable bound for the size of $E_T(\mathbf{Q})$. The existence of the group structure in an elliptic curve does not depend on the field we are taking coordinates in. So, for instance, if $A, B \in \mathbf{Z}$, then for all primes p , the same equation defining an elliptic curve over \mathbf{Q} defines an elliptic curve over the finite field \mathbf{F}_p . The relationship between these two curves can help us in our purpose, using the next result ([Cassels 1991], [Husemoller 1987]):

Theorem.— Let E be an elliptic curve in Weierstrass form $Y^2 = X^3 + AX + B$, with $A, B \in \mathbf{Z}$. If $p > 2$ is a prime number such that it does not divide Δ , then the mapping

$$\begin{aligned} \text{red}_p : E_T(\mathbf{Q}) &\longrightarrow E(\mathbf{F}_p) \\ (\alpha_1, \alpha_2) &\longmapsto (\overline{\alpha_1}, \overline{\alpha_2}) \\ \mathcal{O} &\longmapsto \mathcal{O} \end{aligned}$$

is an injective group homomorphism (where $\overline{\alpha_i}$ denotes the residue classes of α_i modulo p).

Primes which do not divide Δ are called good primes and the induced group homomorphisms are called good reductions. So, choosing some prime p not dividing Δ and computing how many points lie in $E(\mathbf{F}_p)$ we must obtain a multiple of the order of $E_T(\mathbf{Q})$. Our practical choice has been taking three primes (as small as possible), computing the number of points in each case and finding the greatest common divisor of all those quantities. In most cases, this bound was found to be the actual order of $E_T(\mathbf{Q})$.

There are, however, some cases which does not fit this scheme. For example, the curve defined by $Y^2 = X^3 + X$ has the property that, $E_T(\mathbf{Q}) = C_2$ but, for every good prime p the order of $E(\mathbf{F}_p)$ is divisible by 4.

Trying then to be a bit more accurate, we computed not only the order of $E(\mathbf{F}_p)$, but also how many elements of order 2 it had. So, if $E(\mathbf{F}_p)$ presented more points of order 2 than E itself, our choosing for the bound can be smaller than the order of $E(\mathbf{F}_p)$. In the above example, as E_3 is isomorphic to C_4 and $E(\mathbf{F}_5)$ is isomorphic to $C_2 \times C_2$, the bound actually found is the order of the group $E_T(\mathbf{Q})$.

So, if $|E(\mathbf{F}_p)| = M$, the number of points of order 2 in E is s and the number of points of order 2 in $E(\mathbf{F}_p)$ is t , the choosing of the bound goes like this:

- (a) If $s = t$, we choose M .
- (b) If $(s, t) \in \{(0, 1), (1, 3)\}$ then we choose $M/2$.
- (c) If $(s, t) = (0, 3)$ then we can choose $M/4$ as the bound.

Note that one needs the fact that $E(\mathbf{F}_p)$ is a finite group with, at most, three elements of order 2.

4. Points of given order

We will explain now how to decide when an elliptic curve defined over the rationals has a point of a given order, say n , where $n = 4, \dots, 10, 12$. First we need a result on parametrization of torsion structures. Most cases are proved (quite straightforwardly) in [Husemoller 1987]. Also see [Kubert 1976] for a more exhaustive table, without any proofs.

Theorem.— Every elliptic curve with a point P of order $n = 4, \dots, 9, 10, 12$ can be written in the following Tate normal form

$$Y^2 + (1 - c)XY - bY = X^3 - bX^2,$$

with the following relations:

- (1) If $n = 4$, $b = \alpha$, $c = 0$.
- (2) If $n = 5$, $b = \alpha$, $c = \alpha$.

- (3) If $n = 6$, $b = \alpha + \alpha^2$, $c = \alpha$.
- (4) If $n = 7$, $b = \alpha^3 - \alpha^2$, $c = \alpha^2 - \alpha$.
- (5) If $n = 8$, $b = (2\alpha - 1)(\alpha - 1)$, $c = b/\alpha$.
- (6) If $n = 9$, $c = \alpha^2(\alpha - 1)$, $b = c(\alpha(\alpha - 1) + 1)$.
- (7) If $n = 10$, $c = (2\alpha^3 - 3\alpha^2 + \alpha)/[\alpha - (\alpha - 1)^2]$, $b = c\alpha^2/[\alpha - (\alpha - 1)^2]$.
- (8) If $n = 12$, $c = (3\alpha^2 - 3\alpha + 1)(\alpha - 2\alpha^2)/(\alpha - 1)^3$, $b = c(2\alpha - 2\alpha^2 - 1)/(\alpha - 1)$.

Suppose then that we want to check if a given curve E defined by $Y^2 = X^3 + AX + B$ has a point of order n . Assume it possesses such a point: therefore E must be isomorphic to one curve lying in the one-parameter family. Then we simply compute the Weierstrass normal form of a generic curve in the family and check the conditions given at the end of section 1 for two curves in Weierstrass form to be isomorphic.

Example.— Let us give an example with $n = 5$. Suppose that we would like to know if our curve $Y^2 = X^3 + 12933X - 2285226$ (this is curve 110A1(C) from [Cremona 1992]) has a point of order 5. If it is the case, the curve must be isomorphic, by a linear change of variables, to one lying in the family

$$Y^2 + (1 - \alpha)XY - \alpha Y = X^3 - \alpha X^2.$$

So, taking this general equation to Weierstrass form we obtain an equation which we will note $Y^2 = X^3 + A_5(\alpha)X + B_5(\alpha)$. Should this curve be isomorphic to ours, it must hold

$$\frac{A_5(\alpha)^3}{B_5(\alpha)^2} = \frac{12933^3}{2285226^2},$$

which sums up to an equation in the variable α (in our case, of degree 12).

This equation will be called the final polynomial for $n = 5$. For every root α_0 we have to check if there is some $u \in \mathbf{Q}$ verifying

$$\left\{ u^4 = \frac{A}{A_5(\alpha_0)}, \quad u^6 = \frac{B}{B_5(\alpha_0)} \right\}.$$

If there is then we have a point of order 5, which is easily calculated, as $(0, 0)$ is a point of order 5 in the Tate normal form. If not, then there are no points of order 5 in E .

In our example, the only roots were $-1/10$ and 10 . Besides,

$$A_5(10) = A, \quad B_5(10) = B,$$

so in fact there is a point of order 5 in our curve. Tracing back the changes of variables a point of order 5 turns out to be $(123, 1080)$.

The only remaining case is $n = 3$ that is, we need a procedure for deciding if an elliptic curve has a point of order 3. There is also a Tate normal form for this case, but it has some inconveniences, being the heaviest one that the family of curves depends now on two parameters. However, there is a well-known property which can be used ([Cassels 1966]):

Proposition.— Let E be an elliptic curve given by a Weierstrass equation $Y^2 = X^3 + AX + B$. Then E has a point P of order 3 if and only if there is an integral solution to the equation

$$3X^4 + 6AX^2 + 12BX - A^2 = 0.$$

In this case, the solution is the first coordinate of P . In fact, in the cited article one can find polynomials which characterize points of any order. These polynomials become more complicated as the order grows, but they also allow to obtain a obvious procedure for deciding if there is any point of given order.

5. The algorithm

Given an elliptic curve in Weierstrass form $Y^2 = X^3 + AX + B$, in order to find its torsion group we proceed as follows:

Step 1. Compute the number of points with order 2, that is, the rational solutions for $X^3 + AX + B$.

Step 2. Pick the smaller five (for instance) good primes for E and compute a bound M for the torsion as explained above.

Step 3. If the number of rational solutions is either 0 or 1, then for every divisor d of M , apply the procedure described in the previous section to check if there is a point of order d . If this is done in decreasing order, the first affirmative answer gives us the group (which should be isomorphic to C_d) and one generator: either the point which comes from point $(0, 0)$ in Tate normal form for $n = 4, \dots, 10, 12$ or the point directly obtained for $n = 3$.

Step 4. If the number of rational solutions is 3, then apply the same procedure as above for every divisor d of $M/2$. Now the first affirmative answer gives us the group (which must be $C_2 \times C_d$) and a set of generators (the points of order 2 and the point which comes from point $(0, 0)$ in Tate normal form).

6. Explicit calculations

In this section, we will show the computations that led us to the implementation of our algorithm in Maple, currently available by anonymous ftp at `ftp://alg7.us.es/pub/Programs/` (comments in Spanish so far...).

So we fix an elliptic curve E , given by $Y^2 = X^3 + AX + B$ with $A, B \in \mathbf{Z}$ and we want to know if there is a point of order n on it. For all cases (except $n = 3$) we know this implies solving an equation on a parameter α which comes from the parametrizations of Tate normal form.

However, one may find that “classical” parametrizations, though the simplest ones, are not necessarily the most convenient for our purpose. As we will need to compute the rational solutions of a polynomial in $\mathbf{Z}[X]$, which the best parameter is depends heavily on which root finding method is to be used.

Our choice was the algorithm developed in [Loos 1983], so we had to take into account that the complexity of finding the rational roots a polynomial in $\mathbf{Z}[X]$, say $f(X) = \sum a_i X^i$, of degree n , is $O(\log^2 \|f\|)$, where

$$\|f\| = \sum |a_i|,$$

so one may choose a parameter which minimizes $\|f\|$ when f is the final polynomial. Such a parameter will be called a *minimal parameter*.

Case $n = 4$. We will do this in detail. The general equation was

$$Y^2 + XY - \alpha Y = X^3 - \alpha X^2,$$

provided $b \neq 0, -1/16$.

Once it is taken to Weierstrass normal form, it sums up to

$$Y^2 = X^3 + A_4(\alpha)X + B_4(\alpha),$$

where

$$A_4(\alpha) = -432\alpha^2 - 432\alpha - 27, \quad B_4(\alpha) = -3456\alpha^3 + 6480\alpha^2 + 1296\alpha + 54.$$

So the final polynomial for α , $B(\alpha)^2 A^3 - A(\alpha)^3 B^2$ results

$$\begin{aligned} P_4(\alpha) = & 2^{12}3^6\Delta\alpha^6 - 2^{12}3^7(5A^3 - 27B^2)\alpha^5 \\ & + 2^83^7(59A^3 + 459B^2)\alpha^4 + 2^93^611\Delta\alpha^3 \\ & + 2^43^717\Delta\alpha^2 + 2^43^7\Delta\alpha + 3^6\Delta \end{aligned}$$

Our next step is then to find a minimal parameter (that is, a parameter minimizing the norm of its final polynomial). So we find a new parameter $\beta = r\alpha + s$. Obviously we need our new final polynomial, $F_4(\beta)$ to lie in $\mathbf{Z}[X]$ so it is plain that the natural choosing for r must be $1/12$. Then we look for a rational s which minimizes $\|F_4\|$. As F_4 was to lie in $\mathbf{Z}[X]$ the possible denominators were bounded (actually they had to be a divisor of 12). We find a minimum for $s = 1/12$ so we took $\alpha = (\beta + 1)/12$ and

$$\begin{aligned} F_4(\beta) = & \Delta\beta^6 - 6(34A^3 - 135B^2)\beta^5 + 3(851A^3 + 2646B^2)\beta^4 \\ & + 4(313A^3 + 5940B^2)\beta^3 - 6(95A^3 + 2646B^2)\beta^2 \\ & - 24(A^3 - 135B^2)\beta + 49A^3 - 216B^2. \end{aligned}$$

If we set $N = \max\{|A|^3, |B|^2\}$ then

$$\|F_4\| \leq 56667N \simeq 2^83^25^2N.$$

We present below all the minimal parameters along with bounds for the seminorm of the final polynomials, calculated as above.

Case $n = 5$. $\beta = \alpha$, $\deg(F_5) = 12$, $\|F_5\| \leq 898312N \simeq 2^{12}3^25^2N$.

Case $n = 6$. $\alpha = \beta/3 - 1/3$, $\deg(F_6) = 12$, $\|F_6\| \leq 2220071N \simeq 2^43^25^6N$.

Case $n = 7$. $\beta = \alpha$, $\deg(F_7) = 18$, $\|F_7\| \leq 110725743N \simeq 2^{22}3^3N$.

Case $n = 8$. $\alpha = \beta + 1$, $\deg(F_8) = 24$, $\|F_8\| \leq 46702469380N \simeq 2^93^55^8N$.

Case $n = 9$. $\beta = \alpha$, $\deg(F_9) = 36$, $\|F_9\| \leq 11353024920N \simeq 2^{10}3^65^6N$.

Cases $n = 10$ and $n = 12$ can of course be worked out in the same way but the polynomials get quite unpractical. As

$$E_T(\mathbf{Q}) = \begin{cases} C_{10} \iff C_2, C_5 \subset E_T(\mathbf{Q}) \\ C_{12} \iff C_4, C_6 \subset E_T(\mathbf{Q}) \end{cases}$$

there is no necessity of finding the actual polynomials F_{10} and F_{12} . In these cases, the generator can be easily computed using the duplication formula.

The leading coefficient of all final polynomials turns out to be Δ . Indeed, one can look for a parameter such that the leading coefficient and the independent term of its final polynomials are Δ . So, if the factorization of Δ is known, this final polynomials can speed up the process, as all the possible rational roots of the final polynomials are known in advance.

7. Complexity and some examples

As in the previous section, let

$$N = \max \{ |A|^3, |B|^2 \}.$$

We will show that the running time of our algorithm is $\mathcal{O}(K \log^2 N)$ for some $K \in \mathbf{N}$. Unless otherwise stated, [Cohen 1993] is the reference here for the details.

The computation of the points of order two can be clearly accomplished in the expected time, using, for instance, the algorithm given in [Loos 1983]. Note that, should this be the case, it can also be used for checking the existence of points with order three, with the desired complexity.

The bounding of the torsion consists only on arithmetical operations on affine planes \mathbf{F}_p , with p not dividing Δ . It is clear that there are primes smaller than N which not divide Δ . Of course, it is known that arithmetical operations with data bounded by Δ can be carried out in $\mathcal{O}(\log^2 N)$ time.

So it only remains checking step 3 (step 4 is analogous) for the cases $n = 4, \dots, 10, 12$. But note that all the coefficients of our minimal polynomials are bounded by cN , for some natural c . This means that, for a rational root, written in irreducible form $\alpha_0 = \beta_0/\gamma_0$, we have

$$|\beta_0|, |\gamma_0| < cN.$$

Therefore, if we want to find out if there exists some $u \in \mathbf{Q}$ such that $u^4 = A/A_n(\alpha_0)$ we only have to put $A/A_n(\alpha_0)$ in irreducible form (that amounts to find the gcd and divide) and compute the square root of its numerator and denominator twice. All these operations can be carried out in the expected time. If such an u exists, it is just a matter of arithmetical checking seeing if $u^6 = B/B_n(\alpha_0)$.

Some time results of our algorithm are given in the following examples table, using our MapleV routine.

$$\begin{aligned} E_1 : Y^2 &= X^3 - 98D6E49C45C901B \cdot X + B5D1E097F653622F55B036 \\ E_2 : Y^2 &= X^3 - (A_2/A'_2)X + (B_2/B'_2) \\ E_3 : Y^2 &= X^3 - (A_3/16)X - (B_3/32) \end{aligned}$$

where

$$\begin{aligned} A_2 &= 83ACFBAEC1BB1AC8EA33B897FDE9672AB898D04622635/ \\ &198248803F6F6429EC185BB2AB6D5DAE2C41BA0EC07AD5/ \\ &46CFF23FA458FCB36D8E85877CF0 \\ A'_2 &= 4E07B196F78B523E2BF8B93D9FF09BFF22E07284643617/ \\ &AD603BDEBE49E967484527B634E2990C1E19261C903/ \\ &AAC97D0F23EE86534D5011DF9A71 \end{aligned}$$

$$\begin{aligned}
B_2 &= 1594F960645253D0B7F933BFD50446DC3FC067CAFEDB11/ \\
&\quad E76E7EBBDA0FCB2EF4AC34672D4B6469AD156134B7DEA/ \\
&\quad 2FC9C7EFA07084E7695B18DBE22D436EEE2BB5EA14C26/ \\
&\quad D67AB385078CB862970A2B56D62C837D4E00A097490 \\
B'_2 &= AC51A232098DD799F2D035E3B630C2EE79B9C00C70B9013/ \\
&\quad 071A6A0011C7A689A577D55A9BCCDA3FDCCE2FB25958A/ \\
&\quad D9D1F26D9D0D118651B0B5548FF001466E8D0BF7946D23F/ \\
&\quad 9319CE52A96C7C9B2D0E37DEC87027D90109 \\
A_3 &= AF06EC915A7BC47C45CFBFA797633ACE67A79F7B381D29/ \\
&\quad BCCA243AABA230AF5BAD1058D41582134BECCEF3F8DBB \\
B_3 &= 1BDA1A8FE9A5108EA7DB7FB6AE8EB3F7AB45A8D22614B/ \\
&\quad 93FDB39D03E0B8324128145C706768EF5EE5BE37E68F4C \\
&\quad B5BC9EE31CC5B7EDA2C668D5CF0EFE9AA31F0B460EEB
\end{aligned}$$

Curve	Group	Generator	Time
E_1	C_4	$(1C8CFC03, 100F4DC00)$	2.33s
E_2	C_5	$(\lambda_1/\lambda_3^2, \lambda_2/\lambda_3^3)$	10.66s
E_3	$C_2 \times C_4$	(λ_4, λ_5)	3.85s

where

$$\begin{aligned}
\lambda_1 &= -1A8019538D071D5BFD9EEBA7B19BE9124EB6E592F0D15/ \\
&\quad B0DD77D8016A58C \\
\lambda_2 &= -1626E05A34E5EA7E90A84BF3C4D604949BAA0DA532CDE1/ \\
&\quad 147804F9E6491E9E49F16F356882A85DA4C9785AC75C \\
\lambda_3 &= 17C6E3C032F89045AD746684045E05 \\
\lambda_4 &= -7A36225A2ADAAFA9B059FF46EE903619BD0C4E2AD3AA1/4 \\
\lambda_5 &= -897CE6A57036059EE6653F2FCC623CDCF4ADD7F02E202A/8
\end{aligned}$$

The computations have been performed in a KMD300 computer. Note that our current implementation does not include so far the root finding algorithm of [Loos 1983] but Maple V 5.1 built-in routine, so it is hoped that a complete implementation of our algorithm will obtain even better results.

We have compared our algorithm with, probably, the two most efficient current ones: Pari/GP built-in procedure, `elltors` (see [Batut et al. 2000]) and the routine `Tor` from the Maple package APECS (see [Connell 1999]).

Pari/GP `elltors` follows the algorithm described in [Doud 1998], using the analytic parametrization of the curve. It is extremely fast and, besides, the periods of the lattice associated to the curve are directly computed by Pari/GP when you enter the curve with the routine `ellinit`. However, in some cases (we can not figure out when or why), `elltors` needs such a precision that it may become unpractical. It remains, however, as our favourite choosing for medium-size coefficients. Here are the time results, expressed as (time for `ellinit`) + (time for `elltors`), for the previous examples, together with the precision (by 100) required.

Curve	Precision	Time
E_1	> 3600	??
E_2	1300	1.05s + 11.92s
E_3	200	0.06s + 0.08s

For E_3 `elltors` gave an incorrect result: it output C_4 for the structure. Hence there appears to be some minor bug in the implementation. In all our computations, no errors were found in `elltors` when working with cyclic groups.

APECS `Tor` uses the polynomials mentioned at the end of section 4. When you introduce a curve, which you must do before computing its torsion, it computes a great deal of data, in particular a bound for the torsion subgroup and other relevant quantities. If data are moderately large (even significantly smaller than the examples) this takes a huge lot of time: we mean *hours* for the examples above. Anyway, its library is really huge, so, for small-size coefficients, APECS will surely have a lot of information (of course everything concerning rational torsion points) only to look up to.

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