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Abstract

A combinatorial description of the minimal free resolution of a lattice ideal allows to the connection of Integer Lineal Programming and Algebra. The non null reduced homology spaces of some simplicial complexes are the key. The extremal rays of the associated cone reduce the number of variables.

Introduction

The objective of this paper is to describe how Integer Lineal Programming allows us to obtain the minimal free resolution of a lattice ideal, I, from the generators of the semigroup, S, which parametrizes the associated algebraic variety.

Concretely, Hilbert bases of some diophantine systems are employed. These bases are the solution of the typical Integer Lineal Programming Problem, but where the minimality respect to a cost map is not imposed.

Anybody who has solved linear diophantine equations in non negative integers, even with the more recent methods (see [19], [21], [24], [42] and [44]), knows that only in the case of a few variables the problem is tractable. It is well-known that this problem is NP-complete (see for example [36]). Therefore, from the computational viewpoint, our description is not practical in order to obtain the minimal free resolution. However, the method can be used to the contrary. Our description allows the understanding of the relation between the syzygies of the ideal and Integer Lineal Programming. One can compute with Gröbner bases using for example the Schreyer Theorem and its improvements (see [33]), and look for applications to Integer Programming. This philosophy comes from [20] and [44], and provides a lot of applications in [48]. However, at the moment only the case of the ideal I has been employed, but not the syzygies of the higher order (the ideal can be considered as the syzygies of order zero). Our description yields a possible way to attempt a generalization.

As in [30] and [46], the combinatorial objects we use are simplicial complexes. Concretely, for any element of the semigroup S, we associate two simplicial complexes. The elements in the semigroup represent the degrees of the syzygies, in fact, the minimal free resolution is S-graded. The study of the non null reduced homology spaces of the simplicial complexes provides the concept of

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i-triangulation. This concept is the key in order to understand the relation between Integer Lineal Programming and Algebra, concretely, between Hilbert bases and ith syzygies.

By means of a partition of the generating set of S, the number of variables is reduced to the number of extremal rays of the associated cone. This is another possible point to continue researching. A generator over each extremal ray is chosen. Fixing the attention on this subset of generators, a new resolution is considered, the minimal free resolution of I over a polynomial ring with only the variables corresponding to these generators.

We begin in section 1 by introducing the algebraic objects we employ. In section 2 and 3 we give the combinatorial description of the two minimal free resolutions respectively. In section 4, the relation between the two resolutions is studied. Section 5 is dedicated to the i-triangulations in a simplicial complex. The exposition of how to compute both resolutions with Gröbner bases is in section 6. All these sections include the results we have already obtained using the techniques this paper describes. For details the reader may also want to consult the reference joined to the concrete result. Finally, in section 7, following [4], we introduce another free resolution of the ideal I, *the hull resolution*. The comparison of this resolution with the minimal one is also a research objective.

Another possible application of our description is in Toric Geometry. The normal toric varieties [27], and more generally, the non-normal toric varieties [28] and [48], appear as algebraic varieties whose ideals are lattice ones. Among our results can be found descriptions of the regularity of these ideals as well as upper bounds for the degree of their generators. It is expected that there is some relation between these results and the conjetures of [25] and [48] (see also [47]). For a survey of the modern developments in the theory of toric varieties see [22]. Some applications of this theory to the Arithmetic and Integer Programming can be found in [18].

On the other hand, it is known that any binomial ideal is an intersection of *cellular ideals* [26]. The cellular ideals are closely related to the lattice ideals. Using the cellular decomposition of a binomial ideal, it is possible to obtain information about the binomial ideal from the properties of the lattice ideals (for example, primary decomposition or nilpotence index, see [34] and [35]).

1 The two minimal free resolutions associated with a lattice ideal

Let k be a commutative field and $k[\mathbf{X}] = k[X_1, \ldots, X_n]$ the polynomial ring in n indeterminates, and the ideal $\mathfrak{m} = (X_1, \ldots, X_n)$.

Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice. The *ideal of the lattice* $\mathcal{L}$ is

$$I_{\mathcal{L}} = \langle \mathbf{X}^{u^+} - \mathbf{X}^{u^-} \ \mid \ \mathbf{u} \in \mathcal{L} \rangle,$$

where $u = u^+ - u^-$, $u^+, u^- \in \mathbb{N}^n$, have disjoint support.

Let S be a cancellative commutative semigroup, with zero element and generated by n elements $\Lambda = \{m_1, \dots , m_n\}$. Thus, S is a subsemigroup of a finitely generated abelian group. Denote $G(S)$ the smallest group containing S. The semigroup k-algebra is $k[S] = \bigoplus_{m \in S} k\chi^m$, $(\chi^m \cdot \chi^{m'} = \chi^{m+m'})$. The ideal of S relative to Λ is $\ker(\varphi_0)$, where φ_0 is the k-algebra

$$\varphi_0 : k[\mathbf{X}] \longrightarrow k[S]$$

defined by $\varphi_0(X_i) = \chi^{m_i}$. Notice that φ_0 is surjective, and hence $k[S] \simeq k[\mathbf{X}]/\ker(\varphi_0)$.

If $I_{\mathcal{L}}$ is the ideal of the lattice $\mathcal{L} \subset \mathbb{Z}^n$, then $I_{\mathcal{L}}$ is the ideal of the subsemigroup of $\mathbb{Z}^n/\mathcal{L}$ generated by $\{e_1 + \mathcal{L}, \dots, e_n + \mathcal{L}\}$, where the e_i's are the unit vectors.

On the other hand, the ideal of any semigroup S relative to a generating set Λ is the ideal of the lattice $\{u = (u_1, \dots , u_n) \in \mathbb{Z}^n | \sum u_i m_i = 0\}$. (See [50] for details)

From now on, we fix a lattice $\mathcal{L}$ or equivalently a semigroup S. Assume that $\mathcal{L} \cap \mathbb{N}^n = (0)$, or equivalently $S \cap (-S) = (0)$. Let I be the ideal relative to a fix $\Lambda = \{m_1, \dots , m_n\}$ a generating set of S. Notice that I is S-graded because φ_0 is an S-graded morphism of degree zero, considering $k[S]$ with the natural S-grading and $k[\mathbf{X}]$ as an S-graded ring, assigning the degree m_i to X_i. The condition $S \cap (-S) = (0)$ says that $k[S]_m$, the homogeneous elements of degree $m \in S$ in $k[S]$, is a k-vector space of finite dimension (see [8]).

Another application of the condition $S \cap (-S) = (0)$, is Nakayama's lemma for S-graded $k[\mathbf{X}]$-modules (see [8]). Thus, there exists the S-graded free resolution of $k[S]$, unique regarding isomorphisms. We denote such resolution

$$0 \to k[\mathbf{X}]^{b_p} \overset{\varphi_p}{\to} \cdots \to k[\mathbf{X}]^{b_2} \overset{\varphi_2}{\to} k[\mathbf{X}]^{b_1} \overset{\varphi_1}{\to} k[\mathbf{X}] \overset{\varphi_0}{\to} k[S] \to 0,$$

and $N_i = \ker(\varphi_i)$ the ith module of syzygies $0 \le i \le p$ $(N_0 = I)$.

Notice that

$$b_{i+1} = \dim(N_i/\mathfrak{m}N_i),$$

where $N_i/\mathfrak{m}N_i$ is considered as a k-vector space. Moreover, since this space is S-graded, if $V_i(m) := (N_i/\mathfrak{m}N_i)_m$, where $m \in S$, then

$$b_{i+1} = \sum_{m \in S} \dim V_i(m).$$

The Auslander-Buchbaum theorem guarantees that

$$p = n - \mathrm{depth}_{k[\mathbf{X}]} k[S],$$

where $\mathrm{depth}_{k[\mathbf{X}]} k[S]$ is the depth of $k[S]$ as $k[\mathbf{X}]$-module. It is known that $\mathrm{depth}_{k[\mathbf{X}]} k[S]$ is bounded by $\dim k[S]$, which is the rank of the abelian group $G(S)$. In the case the bound is reached, $k[S]$ is a Cohen-Macaulay ring. Thus, this case will be called Cohen-Macaulay case. On the other hand, if $S \neq \{0\}$, it is satisfied that $\mathrm{depth}_{k[\mathbf{X}]} k[S] \geq 1$.

Assume that $\mathrm{rank}(G(S)) = d$, let $V = G(S) \bigotimes_{\mathbb{Z}} \mathbb{Q}$, and let $C(S)$ be the cone generated by the image $\bar{S}$, of S in V. The cone $C(S)$ is strongly convex because $S \cap (-S) = (0)$. Thus, if f is the number of extremal rays of $C(S)$, then $f \geq d$. This implies that there exists a set $E \subset \Lambda$ with $\sharp E = f$, such that $C(E) = C(S)$, where $C(E)$ is the cone in $V(S)$ generated by E. Fix such a set E and $A = \Lambda \setminus E, \sharp A = n - f = r$.

The Apery set Q of S relative to E is defined as

$$Q = \{q \in S \ \ | \ \ q - e \notin S, \ \forall e \in E\}.$$

Denote $k[E]$ the subalgebra of $k[S]$,

$$k[E] = \bigoplus_{m \in S_E} k\chi^m,$$

where S_E is the subsemigroup of S generated by E. Let $k[\mathbf{X}_E]$ the polynomial ring in the f indeterminates associated with E. $k[\mathbf{X}_E]$ can be projected over $k[E]$, it is enough to associate to the indeterminate X_i the symbol χ^{m_i}, for any $m_i \in E$.

$k[S]$ is a $k[E]$-module, and therefore also a $k[\mathbf{X}_E]$-module. The set

$$\{\chi^q \ \ | \ \ q \in Q\},$$

is a minimal system of generators of $k[S]$ as $k[E]$-module, and therefore, also as $k[\mathbf{X}_E]$-module. Since $k[\mathbf{X}_E]$ is noetherian, Q is a finite set. Suppose that $\beta_0 = \sharp Q$, $Q = \{q_1, \ldots, q_{\beta_0}\}$, and consider

$$\Phi_0 : k[\mathbf{X}_E]^{\beta_0} \longrightarrow k[S]$$

defined by $\Phi_0(e_i) = \chi^{q_i}, 1 \leq i \leq \beta_0$. We can consider the S-graded minimal resolution of $k[S]$ as $k[\mathbf{X}_E]$-module

$$0 \to k[\mathbf{X}_E]^{\beta_q} \stackrel{\Phi_q}{\to} \cdots \to k[\mathbf{X}_E]^{\beta_2} \stackrel{\Phi_2}{\to} k[\mathbf{X}_E]^{\beta_1} \stackrel{\Phi_1}{\to} k[\mathbf{X}_E]^{\beta_0} \stackrel{\Phi_0}{\to} k[S] \to 0,$$

which is unique except isomorphisms. We denote $M_i = \ker(\Phi_i)$ the ith module of syzygies of $k[S]$ as $k[\mathbf{X}_E]$-module, $0 \leq i \leq q$. As before, by S-graded Nakayama's lemma, we obtain

$$\beta_{i+1} = \sum_{m \in S} \dim W_i(m),$$

where $W_i(m) := (M_i/\mathfrak{m}_E M_i)_m$ is consider as a k-vector space, and $\mathfrak{m}_E$ is the ideal of $k[\mathbf{X}_E]$ generated by the indeterminates of $\mathbf{X}_E$ (X_i such that $m_i \in E$).

Now, we will call the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}]$-module *the long resolution*, and *the short resolution* the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module.

2 Combinatorial description of the long resolution

Assume that $S \neq (0)$, and consider the S-graded minimal free resolution,

$$0 \to k[\mathbf{X}]^{b_p} \stackrel{\varphi_p}{\to} \cdots \to k[\mathbf{X}]^{b_2} \stackrel{\varphi_2}{\to} k[\mathbf{X}]^{b_1} \stackrel{\varphi_1}{\to} k[\mathbf{X}] \stackrel{\varphi_0}{\to} k[S] \to 0.$$

For any $m \in S$ we define the simplicial complex:

$$\Delta_m = \{F \subset \Lambda \quad | \quad m - n_F \in S\},$$

where $n_F = \sum_{m\in F} m$. (These simplicial complexes are inspired in some graphs of [45].) Let $\widetilde{H}_i(\Delta_m)$ be the k-vector space of the ith- reduced homology of Δ_m, and $\tilde{h}_i(\Delta_m) = \dim(\tilde{H}_i(\Delta_m))$.

There exists an effective isomorphism

$$(*) \quad \tilde{H}_i(\Delta_m) \simeq V_i(m),$$

for any $m \in S$ and for any i, $1 \le i \le n-2$, (for details see [15],[17] and [7], or also [1]). These isomorphisms are a bridge between Combinatoric and Algebra. For example, notice that the numbers b_i in the long resolution can be described by the following formula

$$b_{i+1} = \sum_{m\in S} \tilde{h}_i(\Delta_m).$$

Another example, $k[\mathbf{X}]$ is Cohen Macaulay if and only if one has $\widetilde{H}_{n-d}(\Delta_m) = 0$ for every $m \in S$, where $d = \text{rank } G(S)$. If $k[\mathbf{X}]$ is Cohen Macaulay then the Cohen Macaulay type $\tau_{k[\mathbf{X}]}$ of $k[\mathbf{X}]$ is given by

$$\tau_{k[\mathbf{X}]} = \sum_{m\in S} \widetilde{h}_{n-d-1}(\Delta_m).$$

Thus, in particular, $k[\mathbf{X}]$ is Gorenstein if and only if $k[\mathbf{X}]$ is Cohen Macaulay and if $\widetilde{H}_{n-d-1}(\Delta_m) \neq 0$ exactly for one m for which, moreover, one has $\widetilde{h}_{n-d-1}(\Delta_m) = 1$. The formula for $\tau_{k[\mathbf{X}]}$ follows from the fact that $\tau_{k[\mathbf{X}]} = b_{n-d}$ in the Cohen Macaulay case. Moreover, it is possible to generalize the well known characterization of Gorensteiness for numerical semigroups due to Kunz [32]. To state the result, notice that Δ_m makes for $m \in G(S)$. It is clear that for $m \in G(S) - S$, Δ_m is the empty simplicial complex and therefore one has $\tilde{H}_i(\Delta_m) = 0$ for such an m and $i = -1, 0, 1, 2$. Also notice that Δ_0 is the only complex among the Δ_m's with the property $\tilde{H}_{-1}(\Delta_m) \neq 0$ (in fact it is a one dimensional space). Finally set $\tilde{H}_i(\Delta_m) = 0$ for $i \in \mathbb{Z}$, $i < -1$, and $m \in G(S)$. From the symmetry of the graded resolution in the Gorenstein case, if R is Gorenstein and let $m \in S$ be the element such that $\tilde{H}_{n-d-1}(\Delta_m) \neq 0$, then for any couple of elements $m_1, m_2 \in G(S)$ with $m_1 + m_2 = m$ and $i \in \mathbb{Z}$ one has

$$\tilde{H}_i(\Delta_{m_1}) \simeq \tilde{H}_{n-d-i}(\Delta_{m_2}).$$

(See [7] for details).

In the case of a numerical semigroup, if c is the least element, such that $m \in S$ for any $m \geq c$, then $\tilde{H}_i(\Delta_m) = 0$ for any $m \geq c + n_\Lambda - 1$ and any i, because Δ_m is the full simplex. Therefore, if S is symmetric, the above isomorphism implies that the matrix $\{\tilde{H}_i(\Delta_m)\}_{i,m}$ is a symmetric matrix. (This particular case was proved in [15])

Another important application of these isomorphisms is the construction of minimal generating sets of syzygies. Notice that

$$S(i) := \{m \in S \ \mid \ \widetilde{H}_i(\Delta_m) \neq 0\}, \ n-2 \geq i \geq 0,$$

is the set of S-degrees for the minimal i-syzygies. The notherian property guarantees that $S(i)$ is a finite set, therefore the following construction provides a method for computing a minimal generating set of N_i.

CONSTRUCTION:
STEP 1: Compute $S(i)$.
STEP 2: For any $m \in S(i)$, take the images of the elements in a basis for the i-reduced homology space $\tilde{H}_i(\Delta_m)$ by the isomorphism.

Step 1 is completely solved in [12], but the partial solution for $i = 0$ appears in [8], and for $i = 1$ in [41]. Step 2 is solved with an algorithmic method in [7] (Remark 3.6).

The case $i = 0$ corresponds to the ideal $I = N_0$. In this case, step 1 is equivalent to determine the element $m \in S$ such that Δ_m is non-connected. These elements are characterized by the concept of *to be m-isolated* ([17]) given by there arithmetical conditions. Concretely:

Let $m \in S$, and let $B = \{i_1, ..., i_p\} \subset C \subset \Lambda$, $C \neq \Lambda$. We shall say B is m-isolated from $\Lambda - C$ if:

1. It is possible to write

$$m = \sum_{j=1}^{p} \gamma_{i_j} n_{i_j} = \sum_{t \notin C} \rho_t n_t,$$

where γ_{i_j}, $\rho_t \in \mathbb{N}$, and $0 < \gamma_{i_j}$ for any j, $1 \leq j \leq p$.

2. If there exists $m' \in S$ such that it is possible to write

$$m' = \sum_{j=1}^{p} \gamma'_{i_j} n_{i_j} = \sum_{t \notin B} \rho_t n_t,$$

where γ'_{i_j}, $\rho_t \in \mathbb{N}$, $\gamma'_{i_j} \neq 0$, and where there exists $t \notin C$ such that $\rho_t \neq 0$, then

$$(\gamma'_{i_1}, ..., \gamma'_{i_p}) \not< (\gamma_{i_1}, ..., \gamma_{i_p}).$$

3. If $B' = \{l_1, ..., l_s\} \subset B$ and there exists $m' \in S$ such that it is possible to write

$$m' = \sum_{j=1}^{s} \gamma'_{l_j} n_{l_j} = \sum_{t \notin B'} \rho_t n_t,$$

where γ'_{l_j}, $\rho_t \in \mathbb{N}$, and where there exists $t \notin C$ such that $\rho_t \neq 0$, then

$$(\gamma'_{l_1}, ..., \gamma'_{l_s}) \not\leq (\gamma_{l_1}, ..., \gamma_{l_s}).$$

The following theorem is obtained:

Theorem 2.1. *([17]) Let $m \in S$, the following conditions are equivalents:*
1: Δ_m is non-connected ($\tilde{H}_0(\Delta_m) \neq 0$).
2: There exists $C \subset \Lambda$, such that:

- *$C = \cup_{j=1}^{g} T_j$.*
- *T_j is m-isolated from $\Lambda - C$, for any j.*
- *$T_j \cap T_{j+1} \neq \emptyset$, for any j, $1 \leq j \leq g-1$.*

This characterization allows us to find the particular solutions given for few generators in the numerical case in [29] (n=3), [6] and [38] (n=4), and [16] (n=5). Moreover, by means of new combinatorial elements, the theorem yields an algorithm. Concretely, the vertices of some ladders, or equivalently, the Hilbert bases of some diophantine systems are used. (See [8] for details)

The case $i = 1$ is solved in [41] by construction of a finite set containing $S(1)$. This set is obtained after studying the non-null spaces $\tilde{H}_1(\Delta_m) \neq (0)$. The concept of *$F$-cavity* in Δ_m allows us to associate with S some diophantine systems. The Hilbert bases of these systems provide a check finite set. This technique is generalized in [12] for $i \geq 2$. A new concept is necessary, *the i-triangulation in Δ_m.*

3 Combinatorial description of the short resolution

Assume that $S \neq (0)$, and consider the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module

$$0 \to k[\mathbf{X}_E]^{\beta_{f-1}} \overset{\Phi_{f-1}}{\to} \cdots \to k[\mathbf{X}_E]^{\beta_2} \overset{\Phi_2}{\to} k[\mathbf{X}_E]^{\beta_1} \overset{\Phi_1}{\to} k[\mathbf{X}_E]^{\beta_0} \overset{\Phi_0}{\to} k[S] \to 0.$$

We will show how this resolution can be described by means of other simplicial complexes. Concretely, if $m \in S$, let T_m be the simplicial complex

$$T_m = \{F \subset E \quad | \quad m - n_F \in S\}.$$

Denote $\widetilde{H}_i(T_m)$ the ith reduced homology space of the simplicial complex T_m, and let $\tilde{h}_i(T_m) = \dim(\tilde{H}_i(T_m))$. We will prove that there exists an isomorphism

$$(**) \quad \tilde{H}_i(T_m) \simeq W_i(m),$$

for any $m \in S$ and for any i, $1 \le i \le f-2$. For this, let us consider $k[S]$ and $k \simeq k[\mathbf{X}_E]/\mathfrak{m}_E$ as $k[\mathbf{X}_E]$-modules and use the commutativity of the functor Tor, concretely

$$\mathrm{Tor}_{i+1}(k[S],k) \simeq \mathrm{Tor}_{i+1}(k,k[S]).$$

In order to compute the space $\mathrm{Tor}_{i+1}(k[S],k)$ as $k[\mathbf{X}_E]$-module, take the Koszul complex for the regular sequence $\{X_i \mid m_i \in E\}$, which is a exact sequence. For simplicity, we assume that $E = \{m_1,\ldots,m_f\}$.

$$0 \to \bigwedge^{f} k[\mathbf{X}_E]^f \stackrel{d_{f-1}}{\to} \cdots \to \bigwedge^{j+1} k[\mathbf{X}_E]^f \stackrel{d_j}{\to} \bigwedge^{j} k[\mathbf{X}_E]^f \stackrel{d_{j-1}}{\to} \cdots \to k[\mathbf{X}_E]^f \stackrel{d_0}{\to} k[\mathbf{X}_E] \to k \to 0.$$

Here d_j is given by

$$d_j(e_{i_0}\wedge\cdots\wedge e_{i_j}) = \sum_{l=0}^{j}(-1)^l X_l \; e_{i_0}\wedge\cdots\wedge e_{i_{l-1}}\wedge e_{i_{l+1}}\wedge\cdots\wedge e_{i_j}.$$

These homomorphism are S-graded of degree 0 assigning the degree $m_{i_0}+\cdots+m_{i_j}$ to the element $e_{i_0}\wedge\cdots\wedge e_{i_j}$. Tensoring this exact sequence with the $k[\mathbf{X}_E]$-module $k[S]$, we obtain the S-graded Koszul complex

$$0 \to \bigwedge^{f} k[S]^f \to \cdots \to \bigwedge^{j+1} k[S]^f \stackrel{d_j}{\to} \bigwedge^{j} k[S]^f \stackrel{d_{j-1}}{\to} \cdots \to k[S]^f \stackrel{d_0}{\to} k[S] \to k \to 0.$$

The restriction to its degree $m \in S$ is the following complex of finite-dimensional k-vector space

$$\cdots \to \bigoplus_{\substack{F\subset E\\ \sharp F=3}} k[S]_{m-n_F} \to \bigoplus_{\substack{F\subset E\\ \sharp F=2}} k[S]_{m-n_F} \to \bigoplus_{\substack{F\subset E\\ \sharp F=1}} k[S]_{m-n_F} \to k[S]_m \to 0.$$

Notice that this complex can be identified with the augmented oriented chain complex of T_m, because

$$k[S]_{m-n_F} = \begin{cases} k, & \text{if } F\in T_m \\ 0, & \text{otherwise}\end{cases}$$

Thus, we obtain that

$$(\mathrm{Tor}_{i+1}(k[S],k))_m \simeq \widetilde{H}_i(T_m).$$

In order to compute $\mathrm{Tor}_{i+1}(k,k[S])$ as $k[\mathbf{X}_E]$-modules, take the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module. Tensoring with $k \simeq k[\mathbf{X}_E]/\mathfrak{m}_E$ it is obtained

$$0 \to (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_{f-1}} \stackrel{\tilde{\Phi}_{f-1}}{\to} \cdots \to (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_2} \stackrel{\tilde{\Phi}_2}{\to} (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_1} \stackrel{\tilde{\Phi}_1}{\to} (k[\mathbf{X}_E]/\mathfrak{m}_E)^{\beta_0} \to 0.$$

Thus, $(\mathrm{Tor}_{i+1}(k,k[S]))_m \simeq W_i(m)$.

Now it is clear that the isomorphism $(**)$ follows from the commutativity of the functor Tor.

As an application of these isomorphisms, if denote

$$D(i) := \{m \in S \mid \widetilde{H}_i(T_m) \neq 0\},$$

we obtain that

$$\beta_{i+1} = \sum_{i \in D(i)} \widetilde{h}_i(T_m), \ 0 \leq i \leq f-2.$$

Notice that, by the noetherian property, $D(i)$ is finite.

In [10] is shown how the sets $D(i)$ can be obtained generalizing the techniques used for computing $S(i)$ in [12]. This process will be recalled in section 5.

4 Relationship between the two resolutions

The objective of this section is to explain how the simplicial complexes T_m and Δ_m are related, and therefore their reduced homology. To find this relationship, following [14], we need to introduce new combinatorial objects.

- For any $m \in G(S)$ and $l \geq -1$, denote by $C_l(\mathbf{Q}_m)$ the vector space which has the set
$$\{L \subset A \mid \sharp L = l+1, m - n_L \in Q\}$$
as a basis.
- For any chain z in $C_l(\mathbf{Q}_m)$, denote by
$\theta_l(z)$ the projection on $C_{l-1}(\mathbf{Q}_m)$ of the simplicial boundary of z.

$\{C_\bullet(\mathbf{Q}_m), \theta_\bullet\}$ is a chain complex for any m ([14]). To understand better the homology of this complex, consider, for any $m \in S$, the following subset of Σ:

$$\mathbf{K}_m = \{L \in \Delta_m \mid (L \cap E \neq \emptyset) \ \text{or} \ (L \subset A \ \text{ and } \ m - n_L \in S - Q)\}.$$

It is easy to check that $\mathbf{K}_m$ is a simplicial subcomplex of Δ_m, so that one can consider the chain complex $\widetilde{C}_\bullet(\mathbf{K}_m)$ and the relative chain complex $\widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$.

Notice that, by construction, one has an identification $C_\bullet(\mathbf{Q}_m) \simeq \widetilde{C}_\bullet(\Delta_m, \mathbf{K}_m)$. If $m \in Q$, then one has that $C_\bullet(\mathbf{Q}_m) \simeq k$ and $\mathbf{K}_m = \emptyset$. Otherwise, if $m \in S \setminus Q$, since $\exists e \in E$ such that $m - e \in S$, one obtains $L = \{e\} \in \mathbf{K}_m$ and $\mathbf{K}_m \neq \{\emptyset\}$. Therefore, $\widetilde{H}_{-1}(\mathbf{K}_m) = 0$ for any $m \in S$. This allows us to deduce, from the exact sequence of complexes,

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\Delta_m) \to C_\bullet(\mathbf{Q}_m) \to 0,$$

that there is a long exact sequence of homology,

$$... \to H_{l+1}(\mathbf{Q}_m) \to \widetilde{H}_l(\mathbf{K}_m) \to \widetilde{H}_l(\Delta_m) \to H_l(\mathbf{Q}_m) \to ...$$

$$... \to \widetilde{H}_0(\mathbf{K}_m) \to \widetilde{H}_0(\Delta_m) \to H_0(\mathbf{Q}_m) \to \widetilde{H}_{-1}(\mathbf{K}_m) = 0 \to$$
$$\widetilde{H}_{-1}(\Delta_m) \to H_{-1}(\mathbf{Q}_m) \to 0.$$

Now, in order to understand the homology $\widetilde{H}_\bullet(\mathbf{K}_m)$, let us consider the simplicial complex given by the following disjointed union of subsets of Σ:

$$\overline{\mathbf{K}}_m := \mathbf{K}_m \cup \{I \cup J, I \subset A, J \subset E \quad | \quad m - n_I - n_J \notin S \text{ and } m - n_I - e \in S, \forall e \in J\}.$$

Notice that any $I \cup J$ in the second set of the above union is such that the cardinality of J is at least 2. The complex $\overline{\mathbf{K}}_m$ is acyclic, i.e. $\widetilde{H}_l(\overline{\mathbf{K}}_m) = 0$ for any $l \geq -1$ (see Corollary 2.1 in [14]). Thus, the long exact sequence of homology coming from the exact sequence of chain complexes

$$0 \to \widetilde{C}_\bullet(\mathbf{K}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m) \to \widetilde{C}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to 0$$

gives rise to an isomorphism $\rho_{l+1} : \widetilde{H}_{l+1}(\overline{\mathbf{K}}_m, \mathbf{K}_m) \to \widetilde{H}_l(\mathbf{K}_m)$, for every $l \geq -1$.

To study the homology $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ let us consider, the chain of simplicial complexes

$$\mathbf{K}_m = \mathbf{M}_m^{(-1)} \subset \mathbf{M}_m^{(0)} \subset \mathbf{M}_m^{(1)} \subset ... \subset \mathbf{M}_m^{(r)} = \overline{\mathbf{K}}_m$$

where $\mathbf{M}_m^{(i)}$, $-1 \leq i \leq r$, is the simplicial subcomplex of $\overline{\mathbf{K}}_m$ given by:

$$\mathbf{M}_m^{(i)} := \mathbf{K}_m \cup \{L = I \cup J \in \overline{\mathbf{K}}_m \quad | \quad I \subset A, J \subset E, \text{ and } \sharp I \leq i\}.$$

Now, $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m)$ can be computed (see [14]) by means of the long exact sequences

$$... \to \widetilde{H}_l(\mathbf{M}_m^{(j)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(i)}) \to \widetilde{H}_l(\mathbf{M}_m^{(k)}, \mathbf{M}_m^{(j)}) \to ....$$

for $-1 \leq i < j < k \leq r$. In fact, to compute $\widetilde{H}_\bullet(\overline{\mathbf{K}}_m, \mathbf{K}_m) = \widetilde{H}_\bullet(\mathbf{M}_m^{(r)}, \mathbf{M}_m^{(-1)})$, it will be enough to use the above exact sequences for the concrete values of (i, j, k) given by $(-1, 0, 1), (-1, 1, 2), ..., (-1, r-1, r)$, and take into account the following result which is obvious by construction (see proposition 4.3 in [14]).

From these sequences, for any $m \in S$, one has (see Proposition 3.2 in [14]): for any $l \geq -1$ and any i, $0 \leq i \leq r$,

$$\widetilde{H}_{l+1}(\mathbf{M}_m^{(i)}, \mathbf{M}_m^{(i-1)}) \simeq \bigoplus_{I \subset A, \sharp I = i} \widetilde{H}_{l-i}(T_{m-n_I})$$

(in this formula, $\widetilde{H}_{l-i}(T_{m-n_I}) = 0$ if either $l + 1 < i$ or $m - n_I \notin S$).

A first application of above formula is that $S(i) \subset C_i$, where

$$C_i = \{m \in S \mid m = \overline{m} + n_F, \text{ with } \overline{m} \in D(t) \text{ and } F \subset A, \sharp F = i - t, \text{ for some } t \geq -1\}.$$

Therefore, in order to determine the set $S(i)$ it is enough to compute $D(t)$ for any t, $-1 \leq t \leq \min(i, f-2)$. Notice that this result allows us to construct the long resolution from the short one.

Other applications are obtained in the case of the ideal I is homogeneous for the natural grading. This case is called projective case, because the ideal I defines a toric projective variety (see for example [48]).

A characterization of when the ideal I is homogeneous is the following:

Proposition 4.1. *([10]) I is homogeneous for the natural grading if and only if there exists $\mathbf{w} \in \mathbb{Q}^d$ such that $\mathbf{w} \cdot \pi(m_i) = 1$, for any $i = 1, \dots, n$.*

Here we are supposing that

$$G(S) = \mathbb{Z}^d \oplus \mathbb{Z}/a_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_s\mathbb{Z},$$

with $a_i \in \mathbb{Z}$ non null, $1 \le i \le s$, $n = d + s$, and that π is the projection over the first coordinates

$$\pi : \mathbb{Z}^{d+s} \longrightarrow \mathbb{Z}^d.$$

Assume that I is a homogeneous ideal. In this case, it is well defined $||m|| = ||\alpha||_1$, where $m = \sum_{i=1}^n \alpha_i m_i$ and $||\alpha||_1 = \sum_{i=1}^n \alpha_i$.

It is well-known (see, for example, [3]) that the regularity of I is

$$\mathrm{reg}(I) = max_{0 \le i \le n-2}\{t_i - i\},$$

where t_i is the maximum degree of the minimal i-syzygies of I, i.e. $t_i = max\{||m|| \mid m \in S(i)\}$.

Using the exact sequences associated with the filtration $\{M_m^{(j)}\}$, the following formula is obtained.

Theorem 4.2. *([10])*

$$reg(I) = max_{-1 \le i \le f-2}\{u_i - i\},$$

where $u_i = max\{||m|| \mid m \in D(i)\}$.

Therefore, one can read the regularity in the short resolution, it is not necessary to use the large one.

Another result obtained using these techniques is an effective upper bound for the degrees of the equations defining toric projective varieties. Concretely, let $L : S \to \mathbb{N}$ be the map defined by $L(m) = ||m||$. For any $t \ge 0$, let $H^t := \{m \in S \mid L(m) = t\}$, and denote

$$Q^t := Q \cap H^t,$$

and

$$t_0 := min\{t \mid Q^t = \emptyset\}.$$

On the other hand, from the above sequences, making substitutions of the formulas of homology, the following map is obtained

$$\bigoplus_{a \in A} \widetilde{H}_0(T_{m-a}) \overset{\varphi_m}{\to} \widetilde{H}_0(T_m).$$

Let

$$t_1 := \min\{t \mid coker(\varphi_m) = 0 \ \ \forall m \in H^t\},$$

i.e. the minimum $t \in \mathbb{N}$ such that φ_m is surjective for every $m \in H^t$.

Theorem 4.3. *([9]) An effective upper bound for the degrees of the polynomials in a minimal generating set of the equations of a toric projective variety is $max(t_0, t_1)$.*

5 i-Triangulations

The objective of this section is to describe how the sets $S(i)$, $0 \leq i \leq n-2$, and $D(i)$, $0 \leq i \leq f-2$, can be obtained solving diophantine systems in non negative integers.

Notice that $D(-1) = Q$, and since $C(E) = C(S)$ for any element $a \in A$ there exists $q_a \in \mathbb{N}$ such that

$$q_a \cdot a = \sum_{e \in E} \lambda_e \cdot e$$

with $\lambda_e \in \mathbb{N}$. Therefore, Q can be obtained checking whether the elements $m = \sum_{a \in A} \lambda_a \cdot a$, with $\lambda_a < q_a$, are in Q.

For solving the other cases, we need the concept of *i-triangulation in a simplicial complex.* Let Δ be an abstract simplicial complex with vertices over a finite set $\mathcal{V}$.

The reduced i-homology of the simplicial complex Δ is the k-vector space

$$\tilde{H}_i(\Delta) = \tilde{Z}_i(\Delta)/\tilde{B}_i(\Delta),$$

where $\tilde{Z}_i(\Delta)$ and $\tilde{B}_i(\Delta)$ are the spaces of cycles and boundaries respectively.

Let $i \geq 0$ and $F \subset \mathcal{V}$. We will say that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F if the following properties are satisfied:

1. $\sharp F_j = i+1$, $\forall j = 1, \ldots, t$.
2. $F = \bigcup_{j=1}^{t} F_j$.

We will say that τ is an i-triangulation of F in Δ, if $F_j \in \Delta$, $\forall j = 1, \ldots, t$, and $F \notin \Delta$.

If $\tilde{H}_i(\Delta) \neq 0$, then there is $c \in \tilde{Z}_i(\Delta) - \tilde{B}_i(\Delta)$, $c = \sum_{j=1}^{t} \lambda_j F_j$, such that $\tau = \{F_1, \ldots, F_t\}$ is an i-triangulation of F in Δ, for $F = \bigcup_{j=1}^{t} F_j$.

In the cases $\Delta = \Delta_m$ ó T_m, $\mathcal{V} = \Lambda$ ó E respectively, if $F \subset \mathcal{V}$, and $\tau = \{F_1, \ldots, F_t\}$ is a i-triangulation of F, in Δ_m ó respectively in T_m, we can associate with τ a diphantine system solution. Concretely, let $\mathcal{G}$ be the matrix whose columns are the chosen generators of S, $\mathcal{G} := (m_1|\ldots|m_n) \in \mathcal{M}_{(d+s)\times n}(\mathbb{Z})$, considering the m_i as elements in $\mathbb{Z}^{d+s}$, and let

$$\mathcal{G}(t) := \begin{pmatrix} \mathcal{G} & -\mathcal{G} & 0 & 0 & 0 & & 0 & 0 \\ 0 & \mathcal{G} & -\mathcal{G} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathcal{G} & -\mathcal{G} & 0 & & 0 & 0 \\ & & \ddots & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & & \mathcal{G} & -\mathcal{G} \end{pmatrix} \in \mathcal{M}_{(d+s)(t-1)\times nt}(\mathbb{Z}).$$

Denote by $e_{F_l} \in \mathbb{N}^n$ the vector with all its coordinates zero except those indicated in the set F_l. Let $e_\tau := (e_{F_1}, \ldots, e_{F_t}) \in \mathbb{N}^{nt}$ and let

$$R_\tau := \{\alpha = (\alpha^{(1)}, \ldots, \alpha^{(t)}) \in \mathbb{N}^{nt} \;\mid\; \mathcal{G}(t)\alpha = 0 \;\;, \alpha \gg e_\tau\},$$

where $\gg$ is the natural partial order in $\mathbb{N}^{nt}$. Since τ is a triangulation of F in Δ_m (respectively T_m), there is $\alpha \in R_\tau$ such that $\mathcal{G}\alpha^{(1)} = \cdots = \mathcal{G}\alpha^{(t)} = m \in S$.

Notice that R_τ doesn't depend on m. Therefore, given $F \subset \mathcal{V}$, and $\tau = \{F_1, \ldots, F_t\}$ a i-triangulation of F, we can consider the set R_τ and the set

$$\Sigma R_\tau := \{m \in S \ \mid \ m = \mathcal{G}\alpha^{(1)}, \text{ for any } \ \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in R_\tau\}.$$

The elements $m \in S$ such that τ is i-triangulation of F in Δ_m ó respectively in T_m are in ΣR_τ. We need to be more precise and to give a finite subset of ΣR_τ with the same property. For this, we consider

$$\mathcal{H}R_\tau := \{\alpha \in R_\tau \ \mid \ \alpha \text{ is minimal for } \ll\},$$

which is finite, and

$$\Sigma\mathcal{H}R_\tau := \{m \in S \ \mid \ m = \mathcal{G}\alpha^{(1)}, \text{ for any } \ \alpha = (\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(t)}) \in \mathcal{H}R_\tau\}.$$

In the case of the complexes Δ_m, one has that ([40])

$$S(i) \subset \bigcup_{F \subset \Lambda, \ \sharp F \geq i+2} \ \bigcup_\tau \Sigma\mathcal{H}R_\tau.$$

In the case of the complexes T_m, we also need to consider the set Q. In both cases, there is a partial order which refines the final result. The order is different in each case. Concretely:

- $m >_S m'$ if $m - m' \in S$.
- $m >_Q m'$ if $m - m' \in S \setminus Q$.

Moreover, if $H \subset S$, we will say that $m \in H$ is $S-$minimal (resp. $Q-$minimal) in H if $m \geq_S m'$ (resp. $m \geq_Q m'$), with $m' \in H$, implies that $m = m'$.

Let

$$C_\tau := \{m \in \Sigma R_\tau \ \mid \ m \text{ is } S\text{-minimal in } \Sigma R_\tau\},$$

and $M_\tau := \{m \in \Sigma R_\tau \ \mid \ m \text{ is } Q-\text{minimal in } \Sigma R_\tau\}$.

C_τ is finite because $C_\tau \subset \Sigma\mathcal{H}R_\tau$ (see [12] for details). M_τ is finite because $M_\tau \subset \Sigma\mathcal{H}R_\tau + Q$ (see [10] for details).

The relation of the elements in $S(i)$ (resp. $D(i)$) and C_τ (resp. M_τ) is the following: If $m \in S(i)$ (resp. $m \in D(i)$), then there exists $\tau = \{F_1, \ldots, F_t\}$ $i-$triangulation of $F = \cup_{j=1,\ldots,t} F_j$ such that $m \in C_\tau$ (resp. $m \in M_\tau$). (See [12] and resp. [10] for details.)

Thus, if

$$C'_\tau := \{m \in C_\tau | F \notin \Delta_m\}, \text{ and } C_i(F) := \bigcup_\tau C'_\tau,$$

and if

$$M'_\tau := \{m \in M_\tau | F \notin T_m\}, \text{ and } M_i(F) := \bigcup_\tau M'_\tau,$$

we obtain the following theorem which provides an algorithm to compute the sets $S(i)$ and $D(i)$ (see [12] and [10] for details).

Theorem 5.1. *The elements $m \in S$ such that the simplicial complex Δ_m, respectively T_m, has ith reduced homology non null can be determined solving diophantine systems. Concretely,*

- $S(i) \subset \bigcup_{F \subset \Lambda,\ \sharp F \geq i+2} C_i(F)$, $0 \leq i \leq n-2$.
- $D(i) \subset \bigcup_{F \subset E,\ \sharp F \geq i+2} M_i(F)$, $0 \leq i \leq f-2$.

As an application, we obtain an explicit bound for the degree of the ith minimal syzygies.

Proposition 5.2. *([11]) Let $m \in S$ be the degree of a ith minimal syzygy, $0 \leq i \leq n-2$. There is x, such that $m = \mathcal{G}x$ with*

$$||x||_1 \leq r\,(1 + 2\max|a_j| + ||\mathcal{G}||)^{(d+s)} + (1 + 2\max|a_j| + 4||\mathcal{G}||)^{(d+s)(c-1)} + (i+1)(c+1) - 1,$$

where $c = \begin{pmatrix} f \\ \lfloor f/2 \rfloor \end{pmatrix}$, *and* $||\mathcal{G}|| := sup_l \sum_j |g_{lj}|$.

Moreover, in the homogeneous case, the regularity is bounded.

Theorem 5.3. *([11])*

$$reg(I) \leq r\,(1 + 2\max|a_j| + ||\mathcal{G}||)^{(d+s)} + (1 + 2\max|a_j| + 4||\mathcal{G}||)^{(d+s)(c-1)} + (f-1)(c-1).$$

Notice that these bounds are singly-exponential in the number of extremal rays. Therefore, they are an improvement of the well-known singly-exponential in the number of generators given in [49].

6 Computing with Gröbner Bases

As was explained in section 2, the method porpoused in section 5 for computing the sets $S(i)$, allows us to obtain the minimal free resolution of $k[S]$ as $k[\mathbf{X}]$-module, which we have called the long resolution. In [51] are some explicitied examples.

Using the sets C_i defined in section 4, it is possible to change slightly the method and computing the sets $S(i)$ from the sets $D(t)$ with $-1 \leq t \leq i$ (see [10] for details). The advantage of this change appears when the cardinality of E is strictly less than the cardinality of Λ. In this case, the number of diophantine systems which one must solve decreases. However, even with this improvement, the obtained method is not faster than the method which employs Gröbner bases.

To compute the long resolution of $k[S]$ using Gröbner bases, one must begin computing the ideal I. There exist several methods to do this. Two of them, [23] and [31], are an improvement on the usual method using Elimination Theory (see [48]). These papers consider only the free torsion case. The generalization to non trivial torsion appears in [50]. The advantage of these methods is that they do not need to add new variables like the Elimination theory requires for this concrete problem. An application of these methods is the computing of Hilbert bases of diophantine equations (see [42]), even in the case of equations with congruences (see [41]).

Once a generating set of I is obtained, $\{f_1, \ldots, f_r\}$, we can consider the morphism of free $k[\mathbf{X}]$-modules

$$\varphi : k[\mathbf{X}]^r \longrightarrow k[\mathbf{X}],$$

defined by $\varphi(e_i) = f_i$.

Using the Schreyer Theorem and its improvements (see for example [33]), we can obtain a generating set of $ker(\varphi)$, $\{F_1, \ldots, F_s\}$. This method consists of computing a Gröbner basis.

Notice that the generating set $\{f_1, \ldots, f_r\}$ is minimal if and only if there is no coordinate of F_i in k, for any i, $1 \leq i \leq s$. Moreover, in the case $\{f_1, \ldots, f_r\}$ non minimal, we can remove the redundant elements, using the relations given by the F_i which have a coordinate in k.

Therefore, we can suppose that $\{f_1, \ldots, f_r\}$ is a minimal generating set of I. Thus, $r = b_1$, $\varphi = \varphi_1$ and N_1 is generated by $\{F_1, \ldots, F_s\}$.

Now, we can consider the morphism of free $k[\mathbf{X}]$-modules

$$\varphi' : k[\mathbf{X}]^s \longrightarrow k[\mathbf{X}]^{b_1},$$

defined by $\varphi'(e_i) = F_i$. Using again the Schreyer Theorem, we can compute a generating set of $ker(\varphi')$. Similar reasonings to the previous case yield a minimal generating set of N_1 contained in $\{F_1, \ldots, F_s\}$. This way, we can compute the long resolution. In fact, some Formal Calculus Systems, as Macaulay2 or Singular, have installed this algorithm, even with some improvements (see [33]). From the long resolution one can read the sets $S(i)$.

In order to compute the S-graded minimal free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module with Gröbner bases, we must begin by computing the Apery set, Q. For details see [39].

Assume, for the sake of simplicity, $E = \{m_1, \ldots, m_f\}$ and $A = \{m_{f+1}, \ldots, m_n\}$.

Fix a total order on the monomials of $k[\mathbf{X}] = k[\mathbf{X}_E, \mathbf{X}_A]$, $X_1 < X_2 < \cdots < X_n$, such that:

1. $\mathbf{X}^\alpha < \mathbf{X}^\beta$, implies $\mathbf{X}^{\alpha+\gamma} < \mathbf{X}^{\beta+\gamma}$, for any α, β and γ;
2. If $f = \sum a_\alpha \mathbf{X}^\alpha \in k[\mathbf{X}]$ has the leading monomial $\mathbf{X}^\beta \notin k[\mathbf{X}_A]$, then $\mathbf{X}^\alpha \notin k[\mathbf{X}_A]$, for any α with $a_\alpha \neq 0$.

For example, we can consider the $lex - inf$ order, which is defined

$$\alpha >_{lex-inf} \beta \Longleftrightarrow \alpha <_{lex} \beta,$$

where lex order is the lexicografic order for $X_1 > \cdots > X_n$.

Any order with these properties is not a well-ordering. However, since there exists only a finite number of monomials os S-degree $m \in S$, a Gröbner basis of I can be computed from any S-graded generating set of I. Assume that Γ is the reduced Gröbner basis of I for such a order. Let $\mathcal{B}$ be the set of monomials $\mathbf{X}_A^\alpha$ which are not divisible by any leading monomial of Γ.

Lemma 6.1. *([39])*

$$Q = \{m \in S \ \mid \ m = \sum_{i=f+1}^{n} \alpha_i m_i, \ \textit{where } \mathbf{X}_A^{\alpha} \in \mathcal{B}\},$$

and in particular, the set $\mathcal{B}$ is finite.

Any element in Γ whose leading monomial $\mathbf{X}_E^v \mathbf{X}_A^u$ has variables in $\{X_i \mid 1 \leq i \leq f\}$ (i.e. $v \neq 0$), is , except sign $\pm$,

$$\mathbf{X}_E^v \mathbf{X}_A^u - \mathbf{X}_E^{v'} \mathbf{X}_A^{u'},$$

where $v' \neq 0$, and $\mathbf{X}_A^u, \mathbf{X}_A^{u'} \in \mathcal{B}$, $u \neq u'$. Suppose that $\mathbf{X}_A^u$ and $\mathbf{X}_A^{u'}$ are associated with the elements q_i and $q_j \in Q$ respectively. We associate to the element in Γ, the element in $k[\mathbf{X}_E]^{\beta_0}$ with all the coordinates equal to zero, except the ith and jth ones, which are $\mathbf{X}_A^u$, and $-\mathbf{X}_A^{u'}$ respectively.

In this way, if l_1 is the number of element in Γ of the above form, we obtain G_i elements in $k[\mathbf{X}_E]^{\beta_0}$, $1 \leq i \leq l_1$. Let $\mathcal{M}$ be the matrix

$$\mathcal{M} = (G_1 | \dots | G_{l_1}).$$

$\mathcal{M}$ defines a morphism of free $k[\mathbf{X}_E]$-modules

$$\Psi_1 : k[\mathbf{X}_E]^{l_1} \longrightarrow k[\mathbf{X}_E]^{\beta_0}.$$

Proposition 6.2. *([39])*

$$coker(\mathcal{M}) \simeq_{k[\mathbf{X}_E]} k[S].$$

Therefore, we obtain the first step of a free resolution of $k[S]$ as $k[\mathbf{X}_E]$-module that is S-graded.

$$k[\mathbf{X}_E]^{l_1} \overset{\Psi_1}{\to} k[\mathbf{X}_E]^{\beta_0} \overset{\Phi_0}{\to} k[S] \to 0.$$

Now, in order to obtain the short resolution, it is enough to apply the Schreyer Theorem as we explain at the beginning of this section. Notice that the sets $D(i)$ can be read from the short resolution.

7 The hull resolution of a lattice ideal

Let $\mathcal{L} \subset \mathbb{Z}^n$ be a lattice, such that $\mathcal{L} \cap \mathbb{N}^n = \{0\}$.

We consider the *lattice* $k[\mathbf{X}]$*-module*

$$M = M_{\mathcal{L}} = k[\mathbf{X}]\{\mathbf{X}^a \ \mid \ a \in \mathcal{L}\} = k\{\mathbf{X}^b \ \mid \ b \in \mathcal{L} + \mathbb{N}^n\},$$

submodule of $k[\mathbf{X}^{\pm}] = k[X_1^{\pm 1}, \dots, X_n^{\pm}]$.

For $a \in \mathbb{Z}^n$ and $t \in \mathbb{R}$ we denote $t^a = (t^{a_1}, \dots, t^{a_n})$.

Fix any real number t larger than $(n+1)! = 2 \cdot 3 \cdots (n+1)$.

Let P_t be the convex hull of the point set $\{t^a \mid \mathbf{X}^a \in M\}$. From the proof of Lemma 2.1 in [4] follows that

$$P_t = conv\{t^a \mid a \in \mathcal{L}\} + \mathbb{R}^n_+.$$

Moreover, the vertices of the polyhedron P_t are precisely the points t^a for which $a \in \mathcal{L}$ (see proof of Proposition 2.2 in [4]).

We denote X the cellular complex ([13])whose facets are the bounded faces of P_t. X doesn't depend of t (Theorem 2.3 in [4]). Since P_t is an unbounded n-dimensional convex polyhedron, the dimension of X is at most $n-1$. X is called the *hull complex* of M in [4], and it is represented by $X = \text{hull}(M)$.

Let F be a nonempty face of X. We identify F with its set of vertices, a finite set of $\mathcal{L}$. Set $m_F := \text{lcm}\{m \in F\}$.

Notice that if $F \in X$, then $F + b \in X$, for any $b \in \mathcal{L}$. We take an incidence function satisfying

$$\varepsilon(F, F') = \varepsilon(F + b, F' + b), \text{ for any } b \in \mathcal{L}.$$

We denote F_X the chain complex of $k[\mathbf{X}]$-modules given by

$$\bigoplus_{\substack{F \in X \\ F \neq \emptyset}} k[\mathbf{X}] \cdot e_F \xrightarrow{\partial} \bigoplus_{\substack{F \in X \\ F \neq \emptyset}} k[\mathbf{X}] \cdot e_F, \quad \partial e_F = \sum_{\substack{F' \in X \\ F' \neq \emptyset}} \varepsilon(F, F') \frac{m_F}{m_{F'}} e_{F'}.$$

This chain complex is a free resolution of M (Theorem 2.5 in [4]), called the *hull resolution* of M.

We define the following equivalence relation between the faces of X:

$$F \sim F' \text{ if } F' = F + b \text{ for some } b \in \mathcal{L}.$$

If $F \in X$, $F \neq \emptyset$, we choose $\text{Rep}(F)$ one face in X such that $F \sim \text{Rep}(F)$ and $0 \in \text{Rep}(F)$.

Let

$$\text{Rep}(X/\mathcal{L}) := \{\text{Rep}(F) \mid F \in X, \ F \neq \emptyset\}.$$

The set $\text{Rep}(X/\mathcal{L})$ is finite because if $\{t^0, t^a\}$ is an edge of X, then a is a primitive vector in $\mathcal{L}$. Recall that if $a \in \mathcal{L}$, $a = a^+ - a^-$ with a^+ and a^- two nonnegative vectors with disjoint support. A nonzero vector $a \in \mathcal{L}$ is called primitive if there is no vector $b \in \mathcal{L} \setminus \{a, 0\}$ such that $b^+ \leq a^+$ and $b^- \leq a^-$. The set

$$\{\mathbf{X}^{a^+} - \mathbf{X}^{a^-} \mid a \in \mathcal{L} \text{ is primitive}\}$$

is called the Graver basis of the ideal $I_{\mathcal{L}}$. It is well-known that the Graver basis of $I_{\mathcal{L}}$, equivalently the set of primitive vector of $\mathcal{L}$, is finite (Theorem 4.7 in [48]).

We consider the $k[\mathbf{X}]$-module morphism

$$\text{Rep} : \bigoplus_{\substack{F \in X \\ F \neq \emptyset}} k[\mathbf{X}] \cdot e_F \longrightarrow \bigoplus_{F \in \text{Rep}(X/\mathcal{L})} k[\mathbf{X}] \cdot e_F,$$

defined by $\mathrm{Rep}(e_F) = e_{\mathrm{Rep(F)}}$.

The chain complex of $k[\mathbf{X}]$-modules given by

$$\bigoplus_{F\in \mathrm{Rep}(X/\mathcal{L})} k[\mathbf{X}]\cdot e_F \xrightarrow{\partial^*} \bigoplus_{F\in \mathrm{Rep}(X/\mathcal{L})} k[\mathbf{X}]\cdot e_F, \quad \partial^*(e_F) = \mathrm{Rep}(\partial e_F),$$

is a free resolution of $k[\mathbf{X}]/I_{\mathcal{L}}$ (It follows from Corollary 3.7 in [4]). This free resolution is called the *hull resolution* of $I_{\mathcal{L}}$.

The hull resolution can be non minimal. For example, if $\mathcal{L}$ corresponds to a monomial curve in $\mathbb{A}^3(k)$, then for Cohen-Macaulay case, the hull resolution is not minimal, and for non Cohen-Macaulay case the hull resolution is the minimal resolution. It is an open problem to classify the numerical semigroups (even with four generators) whose hull resolution is minimal.

However, for some particular classes of lattice $\mathcal{L}$ is proved that the hull resolution is minimal. For example, for generic lattice ideals [37]. This class of lattice ideals are the ideals $I_{\mathcal{L}}$ generated by binomials with full support. Theorem 2.9 in [4] proves in particular that the hull resolution of a generic lattice is the minimal resolution.

Another example is the unimodular Lawrence ideals. The Lawrence ideal associated to a $\mathcal{L}$ is the ideal

$$J_{\mathcal{L}} = (\mathbf{X}^a\mathbf{Y}^b - \mathbf{X}^b\mathbf{Y}^a \ \mid \ a-b\in\mathcal{L}) \subset k[\mathbf{X},\mathbf{Y}],$$

see [48] and [5]. (A combinatorial study of the Lawrence ideals is in [43].)

The ideal $J_{\mathcal{L}}$ is unimodular if for the sublattice $\mathcal{L}$ satisfies the following six equivalent conditions (Theorem 1.2 in [2]):

1. The Lawrence ideal $J_{\mathcal{L}}$ possesses an initial monomial ideal which is radical.
2. Every initial monomial ideal of Lawrence ideal $J_{\mathcal{L}}$ is a radical ideal.
3. Every minimal generator of the $J_{\mathcal{L}}$ is a difference of two squarefree monomials.
4. The lattice $\mathcal{L}$ is the image of an integer matrix B with linearly independent columns, such that all maximal minors of B lie in the set $\{0, 1, -1\}$.
5. The lattice $\mathcal{L}$ is the kernel of an integer matrix A with linearly independent rows, such that all maximal minors of A lie in $\{0, m, -m\}$ for some integer m.
6. The quotient ring $k[\mathbf{X}]/J_{\mathcal{L}}$ is a normal domain.

Theorem 3.8 in [2] shows that the hull resolution of the unimodular Lawrence ideal agrees with the minimal free resolution.

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