PREPUBLICACIONES DE LA FACULTAD DE MATEMÁTICAS

DE LA UNIVERSIDAD DE SEVILLA

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Prepublicación nº 36 (Junio-1997)

Sección Álgebra, Computación, Geometría y Topología

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June, 1997

Introduction

In [5] D. Lazard has used homogenisation of polynomials to compute the initial ideal $\mathrm{gr}(J)$ of an ideal J generated by polynomials. In this paper we introduce a homogenisation process of linear differential operators and we consider "admissible" filtrations on the Weyl algebra, generalising L-filtrations [4]. Using an idea similar to Lazard's, we compute generators of a graded ideal $\mathrm{gr}(I)$ with respect to such filtrations. As is proved in [1], this is a key step to compute the slopes of a $\mathcal{D}$-module.

The Weyl algebra $A_n(\mathbf{K})$ of order n over a field $\mathbf{K}$ is the central $\mathbf{K}$-algebra generated by elements x_i, D_i, $i = 1, \ldots, n$, with relations $[x_i, x_j] = [D_i, D_j] = 0, [D_i, x_j] = \delta_{ij}$. It is naturally filtered by the Bernstein filtration associated to the total degree in the x_i's and D_i's. Now consider the **graded K**-algebra B, generated by x_i, D_i, $i = 1, \ldots, n$, and t with homogeneous relations $[x_i, t] = [D_i, t] = [x_i, x_j] = [D_i, D_j] = 0$, $[D_i, x_j] = \delta_{ij} t^2$. Notice that $A_n(\mathbf{K})$ is the quotient of B by the two-sided ideal, generated by the central element $t - 1$. In fact, this algebra coincides with the Rees algebra associated to the Bernstein filtration of $A_n(\mathbf{K})$. The homogenisation of an element in $A_n(\mathbf{K})$ will be an element in B. The homogenisation process for differential operators we present here has the same formal properties as the usual homogenisation of commutative polynomials, and simplifies, considerably, the one studied in [1]. We establish the validity of the division theorem and Buchberger's algorithm to compute standard bases for the algebra B as in [3].

Sections 1 and 2 are devoted to the notions of admissible filtration and δ-standard basis. Section 3 deals with the main purpose of this paper: homogenisation of differential operators and effective computation of δ-standard bases.

We would like to thank Lê Dũng Tráng for valuable suggestions.

*Supported by DGICYT PB94-1435.

§1 Admissible filtrations on the Weyl algebra

Let $\mathbf{K}$ be a field. Let $A_n(\mathbf{K})$ denote the Weyl algebra of order $n \geq 1$, i.e.

$$A_n(\mathbf{K}) = \mathbf{K}[\underline{x}][\underline{D}] = \mathbf{K}[x_1, \ldots, x_n][D_1, \ldots, D_n],$$
$$[x_i, x_j] = [D_i, D_j] = 0, [D_i, x_j] = \delta_{ij}.$$

Given a non-zero element

$$P = \sum_{\alpha,\beta \in \mathbb{N}^n} a_{\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta \in A_n(\mathbf{K}),$$

we denote by $\mathcal{N}(P)$ its *Newton diagram*:

$$\mathcal{N}(P) = \{(\alpha, \beta) \in \mathbb{N}^{2n} \mid a_{\alpha,\beta} \neq 0\}.$$

DEFINITION 1.1.– *Let* $\mathbf{K}$ *be a field. An* order function *on* $A_n(\mathbf{K})$ *is a mapping* $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ *such that:*

1. $\delta(c) = 0$ *if* $c \in \mathbf{K}$, $c \neq 0$.
2. $\delta(P) = -\infty$ *if and only if* $P = 0$.
3. $\delta(P + Q) \leq \max\{\delta(P), \delta(Q)\}$.
4. $\delta(PQ) = \delta(P) + \delta(Q)$.

REMARK 1.2.– *If* δ *is an order function on* $A_n(\mathbf{K})$, *we have* $\delta(\underline{x}^\alpha \underline{D}^\beta \underline{x}^{\alpha'} \underline{D}^{\beta'}) = \delta(\underline{x}^{\alpha+\alpha'} \underline{D}^{\beta+\beta'})$.

DEFINITION 1.3.– *An order function* δ *on* $A_n(\mathbf{K})$ *is called* admissible *if, for all non-zero* $P \in A_n(\mathbf{K})$, *we have* $\delta(P) = \max\{\delta(\underline{x}^\alpha \underline{D}^\beta) \mid (\alpha, \beta) \in \mathcal{N}(P)\}$.

PROPOSITION 1.4.– *Let* $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ *be an admissible order function. Then the family of* $\mathbf{K}$*–vector spaces*

$$G_\delta^k(A_n(\mathbf{K})) = \{P \in A_n(\mathbf{K}) \mid \delta(P) \leq k\}$$

for $k \in \mathbb{Z}$, *is an increassing exhaustive separated filtration of* $A_n(\mathbf{K})$.

DEFINITION 1.5.– *The filtration* $G_\delta^\bullet$ *will be called the associated filtration to the admissible order function* δ, *or the* δ*-filtration. A filtration on* $A_n(\mathbf{K})$ *associated to an admissible order function will be called an* admissible filtration.

PROPOSITION 1.6.– *Let* $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ *be an admissible order function. Then the mapping* $\Lambda_\delta : \mathbb{N}^{2n} \to \mathbb{Z}$ *defined by* $\Lambda_\delta(\alpha, \beta) = \delta(\underline{x}^\alpha \underline{D}^\beta)$ *is the restriction* *to* $\mathbb{N}^{2n}$ *of a unique linear form on* $\mathbb{Q}^{2n}$*, which we will still denote by* Λ_δ*, with integer coefficients* $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ *satisfying* $p_i + q_i \geq 0$ *for* $1 \leq i \leq n$*.*

PROOF: By 1.2, $\Lambda_\delta(\alpha+\alpha', \beta+\beta') = \Lambda_\delta(\alpha, \beta) + \Lambda_\delta(\alpha', \beta')$ for all $\alpha, \alpha', \beta, \beta' \in \mathbb{N}^n$. So there exists a unique linear form $\Lambda_\delta : \mathbb{Q}^{2n} \to \mathbb{Q}$, with integer coefficients $(p_1, \ldots, p_n, q_1, \ldots, q_n)$, such that $\Lambda_\delta(\alpha, \beta) = \delta(\underline{x}^\alpha \underline{D}^\beta)$ for all $(\alpha, \beta) \in \mathbb{N}^{2n}$. We have $q_i + p_i = \delta(D_i x_i) = \delta(x_i D_i + 1) = \max\{\delta(x_i D_i), \delta(1)\} = \max\{p_i + q_i, 0\}$ for all $i = 1, \ldots, n$. □

PROPOSITION 1.7.– *Let* $\Lambda : \mathbb{Q}^{2n} \to \mathbb{Q}$ *be a linear form with integer coefficients* $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ *satisfying* $p_i + q_i \geq 0$ *for all* $i = 1, \ldots, n$*. Then there exists a unique admissible order function* $\delta_\Lambda : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ *such that* $\delta_\Lambda(\underline{x}^\alpha \underline{D}^\beta) = \Lambda(\alpha, \beta)$ *for all* $(\alpha, \beta) \in \mathbb{N}^{2n}$*.*

PROOF: Let us define $\delta_\Lambda : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ by $\delta_\Lambda(0) = -\infty$ and $\delta_\Lambda(P) = \max\{\Lambda(\alpha, \beta) \mid (\alpha, \beta) \in \mathcal{N}(P)\}$ for all non-zero $P \in A_n(\mathbf{K})$. Then we have:

1. $\delta_\Lambda(c) = \Lambda(\underline{0}, \underline{0}) = 0$ for all $c \in \mathbf{K}$, $c \neq 0$.
2. $\delta_\Lambda(P+Q) = \max \Lambda(\mathcal{N}(P+Q)) \leq \max \Lambda(\mathcal{N}(P) \cup \mathcal{N}(Q)) = \max(\Lambda(\mathcal{N}(P)) \cup \Lambda(\mathcal{N}(Q))) = \max\{\max \Lambda(\mathcal{N}(P)), \max \Lambda(\mathcal{N}(Q))\} = \max\{\delta_\Lambda(P), \delta_\Lambda(Q)\}$.
3. For all i, $1 \leq i \leq n$, we have $\delta_\Lambda(D_i x_i) = \delta_\Lambda(x_i D_i + 1) = \max\{p_i + q_i, 0\} = p_i + q_i = \delta_\Lambda(D_i) + \delta_\Lambda(x_i)$.

Last property implies that $\delta_\Lambda(PQ) = \delta_\Lambda(P) + \delta_\Lambda(Q)$. □

Admissible filtrations cover a wide class of filtrations on the Weyl algebra, as it is showed in the next example.

EXAMPLE 1.8.–
1.- The filtration by the order of the differential operators is the associated filtration to the admissible order function δ_Λ *where* $\Lambda(\alpha, \beta) = |\beta| = \beta_1 + \cdots + \beta_n$*.*
2.- The V*-filtration of Malgrange-Kashiwara with respect to the hypersurface* $x_n = 0$ *is the associated filtration to the admissible order function* δ_Λ *where* $\Lambda(\alpha, \beta) = \beta_n - \alpha_n$*.*
3.- Let $L : \mathbb{Q}^2 \to \mathbb{Q}$ *be a linear form with integer coefficients,* $L(a, b) = ra + sb$*, with* $r \geq 0, s \geq 0$ *and let us denote by* $F_L^\bullet$ *the corresponding* L*-filtration (see [1]). The* L*-filtration is the associated filtration to the admissible order function* δ_Λ *where* $\Lambda(\alpha, \beta) = -s\alpha_n + r\beta_1 + \cdots + r\beta_{n-1} + (r+s)\beta_n$*.*
4.- For each k*,* $1 \leq k \leq n$*, let consider* $r, r_1, \ldots, r_k, s_1, \ldots, s_k$ *non-negative integers. Let us denote by* Λ *the linear form on* $\mathbb{Q}^{2n}$ *defined by* $\Lambda(\alpha, \beta) = -s_1\alpha_1 - \cdots - s_k\alpha_k + (r+s_1)\beta_1 + \cdots + (r+s_k)\beta_k + r\beta_{k+1} + \cdots + r\beta_n$*. The associated filtration to the admissible order function* δ_Λ *coincides with the filtration associated to a multi-filtration* $FV^\bullet$ *(see [7]).*

Example 1.9.– *Let Y be the plane curve $x_1^2 - x_2^3 = 0$. The V-filtration on $A_2(\mathbf{K})$ associated to Y is not an admissible filtration.*

Remark 1.10.– *Let*

$$\Lambda(\alpha,\beta) = p_1\alpha_1 + \cdots + p_n\alpha_n + q_1\beta_1 + \cdots + q_n\beta_n$$

be a linear form on $\mathbb{Q}^{2n}$ with integer coefficients and such that $p_i + q_i \geq 0$ for each $i = 1,\ldots,n$. Let $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ the admissible order function associated to Λ (see prop. 1.7):

$$\delta(P) = \max\{\Lambda(\alpha,\beta) \mid (\alpha,\beta) \in \mathcal{N}(P)\}.$$

If $p_i + q_i > 0$ for each $i = 1,\ldots,n$, then the graded ring $\mathrm{gr}_\delta(A_n(\mathbf{K}))$ associated to the δ-filtration is commutative. Let us consider the commutative ring of polynomials $\mathbf{K}[\chi_1,\ldots,\chi_n,\xi_1,\ldots,\xi_n] = \mathbf{K}[\underline{\chi},\underline{\xi}]$ and the $\mathbf{K}$-algebra homomorphism

$$\phi : \mathbf{K}[\underline{\chi},\underline{\xi}] \longrightarrow \mathrm{gr}_\delta(A_n(\mathbf{K})),$$

who sends the χ_i (resp. the ξ_i) to the $\sigma_\delta(x_i)$ (resp. to the $\sigma_\delta(D_i)$), where σ_δ denote the principal symbol with respect to the δ-filtration. Then ϕ is an isomorphism of graded rings, where $\mathbf{K}[\underline{\chi},\underline{\xi}]$ is graded by

$$\deg(\underline{\chi}^\alpha\underline{\xi}^\beta) = \Lambda(\alpha,\beta).$$

If $p_i + q_i > 0$ for each $i = 1,\ldots,r$, $r < n$, and $p_i + q_i = 0$ for each $i = r+1,\ldots,n$, then the graded ring $\mathrm{gr}_\delta(A_n(\mathbf{K}))$ is non-commutative. Let us consider the commutative polynomial ring $R = \mathbf{K}[\chi_1,\ldots,\chi_r,\xi_1,\ldots,\xi_r]$ and the Weyl algebra of order $n-r$ over R

$$S = R[\chi_{r+1},\ldots,\chi_n,\xi_{r+1},\ldots,\xi_n]$$

with relations

$$[\chi_i, a] = [\xi_i, a] = 0, \quad [\chi_i,\xi_j] = \delta_{ij}$$

for all $i,j = r+1,\ldots,n$ and for all $a \in R$. The ring S is graded by

$$\deg(\underline{\chi}^\alpha\underline{\xi}^\beta) = \Lambda(\alpha,\beta)$$

and there is an isomorphism of graded rings

$$\phi : S \longrightarrow \mathrm{gr}_\delta(A_n(\mathbf{K}))$$

who sends the χ_i (resp. the ξ_i) to the $\sigma_\delta(x_i)$ (resp. to the $\sigma_\delta(D_i)$).

Remark 1.11.– *We can also consider admissible filtrations on the ring*

$$\mathcal{D}_n = \mathbb{C}\{\underline{x}\}[\underline{D}]$$

of the germs at the origin of linear differential operators with holomorphic coefficients on $\mathbb{C}^n$. In this case, admissible order functions $\delta : \mathcal{D}_n \to \mathbb{Z} \cup \{-\infty\}$ come from linear forms $\Lambda : \mathbb{Q}^{2n} \to \mathbb{Q}$ with integer coefficients $(p_1,\ldots,p_n,q_1,\ldots,q_n)$ satisfying $p_i + q_i \geq 0$ and $p_i \leq 0$ for all $1 \leq i \leq n$.

§2 δ-exponents and δ-standard bases in $A_n(\mathbf{K})$

Let fix a well monomial ordering $\prec$ in $\mathbb{N}^{2n}$ and let denote by $\leq$ the usual partial ordering in $\mathbb{N}^{2n}$.

Definition 2.1.– *For each admissible order function $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$, we define the following monomial ordering in $\mathbb{N}^{2n}$:*

$$(\alpha,\beta) \prec_\delta (\alpha',\beta') \Leftrightarrow \begin{cases} \delta(\underline{x}^\alpha \underline{D}^\beta) < \delta(\underline{x}^{\alpha'} \underline{D}^{\beta'}) \\ \text{or} \begin{cases} \delta(\underline{x}^\alpha \underline{D}^\beta) = \delta(\underline{x}^{\alpha'} \underline{D}^{\beta'}) \\ \text{and } (\alpha,\beta) \prec (\alpha',\beta'). \end{cases} \end{cases}$$

Remark 2.2.– *If the order function δ takes integer negative values, the ordered set $(\mathbb{N}^{2n}, \prec_\delta)$ is not well ordered, but the restrictions to the level sets of (α,β) such that $\delta(\underline{x}^\alpha \underline{D}^\beta) = c$ are well ordered.*

From now on $\delta : A_n(\mathbf{K}) \to \mathbb{Z} \cup \{-\infty\}$ will denote an admissible order function.

Definition 2.3.– *Given a non-zero element $P \in A_n(\mathbf{K})$, we define the δ-*exponent *of P by $\exp_\delta(P) = \max_{\prec_\delta} \mathcal{N}(P)$. We also denote by $c_\delta(P)$ the coefficient of the monomial of P corresponding to $\exp_\delta(P)$.*

We have the following classical lemma:

Lemma 2.4.– *Given two non-zero elements P, Q in $A_n(\mathbf{K})$ the following properties hold:*

1. $\exp_\delta(PQ) = \exp_\delta(P) + \exp_\delta(Q)$.
2. *If* $\exp_\delta(P) \neq \exp_\delta(Q)$ *then* $\exp_\delta(P+Q) = \max_{\prec_\delta}\{\exp_\delta(P), \exp_\delta(Q)\}$.
3. *If* $\exp_\delta(P) = \exp_\delta(Q)$ *and if* $c_\delta(P) + c_\delta(Q) \neq 0$ *then* $\exp_\delta(P+Q) = \exp_\delta(P)$ *and* $c_\delta(P+Q) = c_\delta(P) + c_\delta(Q)$.
4. *If* $\exp_\delta(P) = \exp_\delta(Q)$ *and if* $c_\delta(P) + c_\delta(Q) = 0$ *then* $\exp_\delta(P+Q) \prec_\delta \exp_\delta(P)$.

Proof: 1. We can suppose without loss of generality $c_\delta(P) = c_\delta(Q) = 1$. Let us write

$$\exp_\delta(P) = (\alpha_1, \beta_1), \quad \exp_\delta(Q) = (\alpha_2, \beta_2)$$

and

$$P = \underline{x}^{\alpha_1} \underline{D}^{\beta_1} + P', \quad Q = \underline{x}^{\alpha_2} \underline{D}^{\beta_2} + Q'.$$

We have:

$$\mathcal{N}(PQ) \subseteq \mathcal{N}(\underline{x}^{\alpha_1} \underline{D}^{\beta_1} \underline{x}^{\alpha_2} \underline{D}^{\beta_2}) \cup \mathcal{N}(P' \underline{x}^{\alpha_2} \underline{D}^{\beta_2}) \cup \mathcal{N}(\underline{x}^{\alpha_1} \underline{D}^{\beta_1} Q') \cup \mathcal{N}(P'Q').$$

An element of $\mathcal{N}(P'\underline{x}^{\alpha_2}\underline{D}^{\beta_2}) \cup \mathcal{N}(\underline{x}^{\alpha_1}\underline{D}^{\beta_1}Q') \cup \mathcal{N}(P'Q')$ has the form

$$(\alpha+\gamma-(\beta-\beta'),\beta'+\varepsilon)$$

with $\beta' \leq \beta$, $\beta-\beta' \leq \gamma$ and $(\alpha,\beta) \prec_\delta (\alpha_1,\beta_1), (\gamma,\varepsilon) \preceq_\delta (\alpha_2,\beta_2)$ or $(\alpha,\beta) \preceq_\delta (\alpha_1,\beta_1), (\gamma,\varepsilon) \prec_\delta (\alpha_2,\beta_2)$. By admissibility

$$\delta(\underline{x}^{\alpha+\gamma-(\beta-\beta')}\underline{D}^{\beta'+\varepsilon}) \leq \delta(\underline{x}^{\alpha}\underline{D}^{\beta}\underline{x}^{\gamma}\underline{D}^{\varepsilon}) = \delta(\underline{x}^{\alpha+\gamma}\underline{D}^{\beta+\varepsilon}),$$

and then

$$(\alpha+\gamma-(\beta-\beta'),\beta'+\varepsilon) \preceq_\delta (\alpha,\beta)+(\gamma,\varepsilon) \prec_\delta (\alpha_1,\beta_1)+(\alpha_2,\beta_2).$$

Now

$$\mathcal{N}(\underline{x}^{\alpha_1}\underline{D}^{\beta_1}\underline{x}^{\alpha_2}\underline{D}^{\beta_2}) = \{(\alpha_1+\alpha_2-(\beta_1-\beta'),\beta'+\beta_2) \mid \beta' \leq \beta_1, \beta_1-\beta' \leq \alpha_2\},$$

and the monomial $\underline{x}^{\alpha_1+\alpha_2}\underline{D}^{\beta_1+\beta_2}$ can not be cancelated. So $\exp_\delta(PQ) = (\alpha_1,\beta_1)+(\alpha_2,\beta_2)$.

The proof of properties 2., 3., 4. is straightforward. □

Definition 2.5.– *Given a non-zero left ideal I of $A_n(\mathbf{K})$, we define*

$$\mathrm{Exp}_\delta(I) = \{\exp_\delta(P) \mid P \in I, P \neq 0\}.$$

Definition 2.6.– *Given a non-zero left ideal I of $A_n(\mathbf{K})$, a δ-*standard basis *of I is a family $P_1,\ldots,P_r \in I$ such that*

$$\mathrm{Exp}_\delta(I) = \bigcup_{i=1}^{r} \exp_\delta(P_i) + \mathbb{N}^{2n}.$$

Proposition 2.7.– *Let I be a non-zero left ideal of $A_n(\mathbf{K})$ and let $P_1,\ldots,P_r$ be a δ-standard basis of I. Then $\sigma_\delta(P_1),\ldots,\sigma_\delta(P_r)$ generate the graded ideal $\mathrm{gr}_\delta(I)$.*

Proof: We follow the proof of lemma 1.3.3 in [1]. Let P be a non-zero element of I. We define inductively a family of elements $P^{(s)}$ of I for $s \geq 0$:

- $P^{(0)} := P$,
- $P^{(s+1)} := P^{(s)} - \frac{c_\delta(P^{(s)})}{c_\delta(P_{i_s})}\underline{x}^{\alpha^s}\underline{D}^{\beta^s}P_{i_s}$, where (α^s,β^s) is an element of $\mathbb{N}^{2n}$ such that $(\alpha^s,\beta^s)+\exp_\delta(P_{i_s}) = \exp_\delta(P^{(s)})$,
- $\delta(P^{(s+1)}) \leq \delta(P^{(s)})$ and $\exp_\delta(P^{(s+1)}) \prec_\delta \exp_\delta(P^{(s)})$.

By the remark 2.2, there is an s such that $\delta(P^{(s+1)}) < \delta(P^{(s)})$. Let s be the smallest integer having this property. Then

$$\sigma_\delta(P) = \sum_{j=0}^{s} \sigma_\delta \left(\frac{c_\delta(P^{(j)})}{c_\delta(P_{i_j})} \underline{x}^{\alpha^j} \underline{D}^{\beta^j} \right) \sigma_\delta(P_{i_j}).$$

□

EXAMPLE 2.8.– *In general, a δ-standard basis of an ideal $I \subseteq A_n(\mathbf{K})$ is not a system of generators of I. For example, take the admissible order function defined by*

$$\delta(\underline{x}^\alpha \underline{D}^\beta) = \beta_n - \alpha_n$$

associated to the V-filtration with respect to $x_n = 0$, and take $I = A_n(\mathbf{K})$, $P = 1 + x_n^2 D_n$. It is clear that $\sigma_\delta(P) = 1$ and then P is a δ-standard basis of I, but obviously P does not generate $I = A_n(\mathbf{K})$ (P is not a unit).

REMARK 2.9.– *Using the isomorphisms ϕ of remark 1.10, one can define, in the obvious way, the Newton diagram $\mathcal{N}(H)$ for each non-zero element $H \in \mathrm{gr}_\delta(A_n(\mathbf{K}))$, and so the δ-exponent $\exp_\delta(H) \in \mathbb{N}^{2n}$, the set $\mathrm{Exp}_\delta(J) \subseteq \mathbb{N}^{2n}$ for each non-zero left ideal $J \subseteq \mathrm{gr}_\delta(A_n(\mathbf{K}))$ and the notion of δ-standard basis for a such J.*
If H is a non-zero homogeneous element in $\mathrm{gr}_\delta(A_n(\mathbf{K}))$, then the exponent of H with respect to $\prec$, $\exp_\prec(H)$, coincides with $\exp_\delta(H)$, and for each non-zero $P \in A_n(\mathbf{K})$ we have

$$\exp_\delta(P) = \exp_\prec(\sigma_\delta(P)).$$

In fact, the notions of δ-standard basis and of $\prec$-standard basis for a non-zero homogeneous left ideal of $\mathrm{gr}_\delta(A_n(\mathbf{K}))$ coincide.
One can show easily that, for a non-zero left ideal $I \subseteq A_n(\mathbf{K})$, a family of elements $P_1, \ldots, P_r \in I$ is a δ-standard basis of I if and only if $\sigma_\delta(P_1), \ldots, \sigma_\delta(P_r)$ is a $\prec$-standard basis of $\mathrm{gr}_\delta(I)$.
As in [3], we have a division algorithm in $\mathrm{gr}_\delta(A_n(\mathbf{K}))$ with respect to the well ordering $\prec$. As a consequence, a $\prec$-standard basis of $\mathrm{gr}_\delta(I)$ is a system of generators of this ideal. This precises the proposition 2.7.

§3 Homogenisation

In this section we denote by A_n the Weyl algebra $A_n(\mathbf{K})$. Let $A_n[t]$ denote the algebra

$$A_n[t] = \mathbf{K}[t, \underline{x}][\underline{D}] = \mathbf{K}[t, x_1, \ldots, x_n][D_1, \ldots, D_n]$$

with relations

$$[t, x_i] = [t, D_i] = [x_i, x_j] = [D_i, D_j] = 0, [D_i, x_j] = \delta_{ij} t^2.$$

The algebra $A_n[t]$ is graded, the degree of the monomial $t^k \underline{x}^\alpha \underline{D}^\beta$ being $k+|\alpha|+|\beta|$.

LEMMA 3.1.– *The* $\mathbf{K}$*–algebra* $A_n[t]$ *is isomorphic to the Rees algebra associated to the Bernstein filtration of* A_n*. The algebra* $\mathbf{K}[t]$ *is central in* $A_n[t]$ *and the quotient algebra* $A_n[t]/\langle t-1\rangle$ *is isomorphic to* A_n*.*

PROOF: Let $B^\bullet$ be the Bernstein filtration of A_n. We have, for each $m \in \mathbb{N}$,

$$B^m(A_n) = \{ \sum_{|\alpha|+|\beta|\leq m} p_{\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta \,|\, p_{\alpha,\beta} \in \mathbf{K}\}.$$

Let

$$\mathcal{R}(A_n) = \bigoplus_{m\geq 0} B^m(A_n)\cdot u^m$$

be the Rees algebra of A_n. We observe that the $\mathbf{K}$-linear map $\phi : A_n[t] \to \mathcal{R}(A_n)$ defined by

$$\phi(t) = u, \phi(x_i) = x_i \cdot u, \phi(D_i) = D_i \cdot u$$

is an isomorphism of graded algebras. □

Given $P = \sum_{\alpha,\beta} p_{\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta$ in A_n we denote by $\mathrm{ord}^T(P)$ its total order

$$\mathrm{ord}^T(P) = \max\{|\alpha| + |\beta| \mid p_{\alpha,\beta} \neq 0\}.$$

DEFINITION 3.2.– *Let* $P = \sum_{\alpha,\beta} p_{\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta \in A_n$*. Then, the differential operator*

$$h(P) = \sum_{\alpha,\beta} p_{\alpha,\beta} t^{\mathrm{ord}^T(P)-|\alpha|-|\beta|} \underline{x}^\alpha \underline{D}^\beta \in A_n[t]$$

is called the homogenisation *of* P*. If* $H = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} t^k \underline{x}^\alpha \underline{D}^\beta$ *is an element of* $A_n[t]$ *we denote by* $H_{|t=1}$ *the element of* A_n *defined by* $H_{|t=1} = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} \underline{x}^\alpha \underline{D}^\beta$*.*

LEMMA 3.3.– *For* $P, Q \in A_n[t]$ *we have:*

1. $h(PQ) = h(P)h(Q)$*.*
2. *There exist* $k, l, m \in \mathbb{N}$ *such that* $t^k h(P+Q) = t^l h(P) + t^m h(Q)$*.*

For any homogeneous element $H \in A_n[t]$ *there exists* $k \in \mathbb{N}$ *such that* $t^k h(H_{|t=1}) = H$*.*

PROOF: 1. We have

$$h(D_i x_i) = h(x_i D_i + 1) = x_i D_i + t^2 = D_i x_i = h(D_i)h(x_i), \quad i = 1, \ldots, n.$$

From this we obtain easily 1.. To prove 2. let us denote $b = \mathrm{ord}^T(P), c = \mathrm{ord}^T(Q), d = \mathrm{ord}^T(P+Q)$ and $e = \max\{b, c\}$. We have: $t^{e-d}h(P+Q) = t^{e-b}h(P) + t^{e-c}h(Q)$.

Let H be a non-zero homogeneous element in $A_n[t]$. Let k be the greatest integer such that t^k divides H. There exists a homogeneous element $G \in A_n[t]$ such that $H = t^k G$ and such that the degree of G is equal to $\mathrm{ord}^T(G_{|t=1})$. We have $H_{|t=1} = G_{|t=1}$ and $t^k h(G_{|t=1}) = t^k G = H$. □

Let fix a well monomial ordering $\prec$ in $\mathbb{N}^{2n}$ and an admissible order function $\delta : A_n \to \mathbb{Z} \cup \{-\infty\}$.

We consider on $\mathbb{N}^{2n+1}$ the following total ordering, denoted by $\prec_\delta^h$, which is a well monomial ordering:

$$(k,\alpha,\beta) \prec_\delta^h (k',\alpha',\beta') \Longleftrightarrow \begin{cases} k+|\alpha|+|\beta| < k'+|\alpha'|+|\beta'| \\ \text{or } \begin{cases} k+|\alpha|+|\beta| = k'+|\alpha'|+|\beta'| \text{ and} \\ (\alpha,\beta) \prec_\delta (\alpha',\beta') \end{cases} \end{cases}$$

Definition 3.4.– *Let $H = \sum_{k,\alpha,\beta} h_{k,\alpha,\beta} t^k \underline{x}^\alpha \underline{D}^\beta \in A_n[t]$. As in §1 we denote by $\mathcal{N}(H)$ the Newton diagram of H:*

$$\mathcal{N}(H) = \{(k,\alpha,\beta) \in \mathbb{N}^{2n+1} \mid h_{k,\alpha,\beta} \neq 0\}.$$

Definition 3.5.– *Given a non-zero element $H \in A_n[t]$ we define the δ-*exponent *of H by $\exp_\delta(H) = \max_{\prec_\delta^h} \mathcal{N}(H)$. We also denote by $c_\delta(H)$ the coefficient of the monomial of H corresponding to* $\exp_\delta(H)$. *We write* $\exp(H)$ *and* $c(H)$ *when no confusion is possible. If J is a non-zero left ideal of $A_n[t]$ we denote by* $\mathrm{Exp}_\delta(J)$ *the set* $\{\exp_\delta(H) \mid H \in J, H \neq 0\}$.

Lemma 3.6.– *Properties 1-4 from lemma 2.4 hold for δ-exponents of elements in $A_n[t]$. Furthermore, if $P \in A_n$ then $\pi(\exp_\delta(h(P))) = \exp_\delta(P)$, where $\pi : \mathbb{N}^{2n+1} = \mathbb{N} \times \mathbb{N}^{2n} \to \mathbb{N}^{2n}$ is the natural projection, and more generally, if $H \in A_n[t]$ is homogeneous then $\pi(\exp_\delta(H)) = \pi(\exp_\delta(h(H_{|t=1}))) = \exp_\delta(H_{|t=1})$.*

Proof: The proof of the first part is similar to that of lemma 2.4. Last property follows from lemma 3.3. □

Theorem 3.7.– *Let $(P_1,\ldots,P_r)$ be in $A_n[t]^r$. Let us denote by*

$$\begin{aligned} \Delta_1 &:= \exp(P_1) + \mathbb{N}^{2n+1} \\ \Delta_i &:= (\exp(P_i) + \mathbb{N}^{2n+1}) \setminus \bigcup_{j=1}^{i-1} \Delta_j, i = 2,\ldots,r \\ \overline{\Delta} &:= \mathbb{N}^{2n+1} \setminus \bigcup_{i=1}^{r} \Delta_i = \mathbb{N}^{2n+1} \setminus \bigcup_{i=1}^{r} (\exp(P_i) + \mathbb{N}^{2n+1}). \end{aligned}$$

Then, for any $H \in A_n[t]$ there exists a unique element $(Q_1,\ldots,Q_r,R)$ in $A_n[t]^{r+1}$ such that:

1. $H = Q_1P_1 + \cdots + Q_rP_r + R$.

2. $\exp(P_i) + \mathcal{N}(Q_i) \subseteq \Delta_i$ *for* $1 \leq i \leq r$.

3. $\mathcal{N}(R) \subseteq \overline{\Delta}$.

PROOF: The proof is the same as in [3] since $\prec_\delta^h$ is a well monomial ordering. □

LEMMA 3.8.– *Let I be a non-zero left ideal of A_n. We denote by $h(I)$ the homogenized ideal of I, i.e. $h(I)$ is the homogeneous left ideal of $A_n[t]$ generated by the set $\{h(P) \mid P \in I\}$. Then:*

1. $\pi(\mathrm{Exp}_\delta(h(I))) = \mathrm{Exp}_\delta(I)$.

2. *Let $\{P_1, \ldots, P_m\}$ be a system of generators of I. Let $\widetilde{I}$ be the left ideal of $A_n[t]$ generated by $\{h(P_1), \ldots, h(P_m)\}$. Then $\pi(\mathrm{Exp}_\delta(\widetilde{I})) = \mathrm{Exp}_\delta(I)$.*

PROOF: 1. Let P be a non-zero element of I. Then the equality $\exp_\delta(P) = \pi(\exp_\delta(h(P)))$ shows that $\mathrm{Exp}_\delta(I) \subseteq \pi(\mathrm{Exp}_\delta(h(I)))$. Let H a non-zero element of the homogeneous ideal $h(I)$. We can suppose H homogeneous. There exist $B_1, \ldots, B_m \in A_n[t]$ and $P_1, \ldots, P_m \in I$ such that $H = \sum_i B_i h(P_i)$. Then, $H_{|t=1} = \sum_i B_{i|t=1} P_i$ belong to the ideal I. The inclusion $\pi(\mathrm{Exp}_\delta(h(I))) \subseteq \mathrm{Exp}_\delta(I)$ follows from 3.6.
2. Write $P = \sum_i C_i P_i \in I$, where $C_i \in A_n$. From lemma 3.3 there exists $k \in \mathbb{N}$ such that $t^k h(P) \in \widetilde{I}$. The equality $\exp_\delta(P) = \pi(\exp_\delta(t^k h(P)))$ shows that $\mathrm{Exp}_\delta(I) \subseteq \mathrm{Exp}_\delta(\widetilde{I})$. Finally, $\widetilde{I} \subseteq h(I)$ implies $\mathrm{Exp}_\delta(\widetilde{I}) \subseteq \mathrm{Exp}_\delta(h(I)) = \mathrm{Exp}_\delta(I)$. □

Let H_1, H_2 be elements of $A_n[t]$. Let us denote $\exp(H_i) = (k_i, \alpha_i, \beta_i)$ and $(k, \alpha, \beta) = l.c.m.\{(k_1, \alpha_1, \beta_1), (k_2, \alpha_2, \beta_2)\}$. There exists $(l_i, \gamma_i, \delta_i)$, for $i = 1, 2$, such that $(k, \alpha, \beta) = (k_1, \alpha_1, \beta_1) + (l_1, \gamma_1, \delta_1) = (k_2, \alpha_2, \beta_2) + (l_2, \gamma_2, \delta_2)$.

DEFINITION 3.9.– *The operator*

$$S(H_1, H_2) = c(H_2) t^{l_1} x^{\gamma_1} \underline{D}^{\delta_1} H_1 - c(H_1) t^{l_2} x^{\gamma_2} \underline{D}^{\delta_2} H_2$$

is called the semisyzygy *relative to* (H_1, H_2).

THEOREM 3.10.– *Let $\mathcal{F} = \{P_1, \ldots, P_r\}$ be a system of generators of a left ideal J of $A_n[t]$ such that, for any (i, j), the remainder of the division of $S(P_i, P_j)$ by $(P_1, \ldots, P_r)$ is equal to zero. Then $\mathcal{F}$ is a δ-standard basis of J.*

PROOF: This theorem is analogous to Buchberger's criterium for polynomials [2]. For example, the proof of [6, Th. 3.3, Chap. 1] can be formally adapted to our case. □

The preceeding theorem gives an algorithm in order to calculate a δ-standard basis of an ideal I of $A_n(\mathbf{K})$ starting from a system of generators: let $\mathcal{F} = \{P_1, \ldots, P_r\}$ be a system of generators of this ideal. We can calculate a δ-standard basis, say $\mathcal{G} = \{G_1, \ldots, G_s\}$, of the ideal $J = \widetilde{I}$ of $A_n[t]$, generated by $\{h(P_1), \ldots, h(P_r)\}$. From 3.8 we have $\pi(\mathrm{Exp}_\delta(\widetilde{I})) = \mathrm{Exp}_\delta(I)$ and then, by 3.6, $\{G_{1|t=1}, \ldots, G_{s|t=1}\}$ is a δ-standard basis of I. Finally, by proposition 2.7, $\{\sigma_\delta(G_{1|t=1}), \ldots, \sigma_\delta(G_{s|t=1})\}$ is a system of generators of $\mathrm{gr}_\delta(I)$.

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