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The standard fan of an analytic $\mathcal{D}$-module

A. Assi, F.J. Castro-Jiménez and M. Granger[†]

Abstract

We extend to the analytic $\mathcal{D}$-module case our results in the algebraic case (see [3]), namely, we associate with any monogeneous module over the ring $\mathcal{D}$ of germs of linear differential operators at the origin of $\mathbf{C}^n$, with holomorphic coefficients, a combinatorial object which we call the standard fan of this $\mathcal{D}$-module (see chapter 6 for a precise geometric description of this object). The main tool of the proof is the homogenization techniques and a convergent division theorem that we prove in the homogenization ring $\mathcal{D}[t]$.

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1 Introduction and notations

The aim of this paper is to associate with any monogeneous module over the ring $\mathcal{D}$ of germs of linear differential operators at the origin of $\mathbf{C}^n$, with holomorphic coefficients, a combinatorial object which we call the standard fan of this $\mathcal{D}$-module. We use here the terminology of Hironaka, the term Gröbner fan being used preferably, in the algebraic-completely algorithmic-case, as in [17] for the commutative analogue. We treated the algebraic case, that is the case of a module over the Weyl algebra in [3]. The standard fan is built in order to take into account all the possible filtrations of the $\mathcal{D}$-module adapted to the given local coordinates, and the variations of the associated graded module.

Our results, while being similar to those in the algebraic case, contain some significant features that are quite different : notably the space $\mathcal{U}$ of allowed filtrations has to be restricted for the filtrations to be of valuation type on the variables x_i; and above all, there is the division theorem, such as we detail below, which is the hard part of the proof.

A variety of similar situations occur in many places in the litterature. Their common point is to involve the restriction of the standard fan to various linear subspaces of $\mathcal{U}$. Some examples are:

In [11], Laumon begins the investigation of a comparison between the V filtration of Malgrange Kashiwara along a hypersurface and the filtration by the order, by developping a theory of filtred $\mathcal{D}$-modules.

In [18], Sabbah and Castro build an analogue of the standard fan associated with the filtration by the order and the filtration V_i of Malgrange-Kashiwara type along the hypersurfaces x_i. This allows Sabbah in [19] to study the specialisations of a $\mathcal{D}$-module $\mathcal{M}$ along these hypersurfaces. Similarly Laurent, in [12] defines the slopes of a $\mathcal{D}$-module along a hypersurface by the jumps in the graded rings of the filtrations interpolating the order filtration F and the V filtration along the given hypersurface.

Let us now detail the content of this paper : we consider the space $\mathcal{U}$ of all linear forms $L : \mathbf{Q}^{2n} \longrightarrow \mathbf{Q}$ compatible with the structure of $\mathcal{D}$ which involves the following two constraints: the filtration by L must induce a series order in restriction to the variables x_i and the weight of $\frac{\partial}{\partial x_i}$ must be greater than the opposite of the (non positive) weight of x_i in view of the compatibility with the commutation rule $[\frac{\partial}{\partial x_i}, a] = \frac{\partial a}{\partial x_i}$. This gives on $\mathcal{U}$ a natural stratification since it is a polyhedral cone with 4^n stratas and a canonical isomorphism type for the ring $\mathrm{gr}^L(\mathcal{D})$ on each stratum.

We prove first that the number of all possible $\mathrm{gr}^L(I)$ for a given ideal I of $\mathcal{D}$ is finite. Our main tool is the homogenization ring, as defined in [10] which we denote by $\mathcal{D}[t]$, and which is isomorphic to the Rees ring $R_F(\mathcal{D})$ with respect to the order filtration F because of the rule $[\frac{\partial}{\partial x_i}, a] = t.\frac{\partial a}{\partial x_i}$. In this ring, we obtain a division

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theorem, which is one of the major difficulties: we use in $\mathcal{D}$ the pseudonorms of the type $||\sum a_{\alpha,\beta}x^\alpha\partial^\beta|| = \sum |a_{\alpha,\beta}|s^{-L(\alpha,\beta)}$ with $s > 0$. We define "elementary division steps" of the type $P = \sum Q_i P_i + R + v(P)$, with v linear automorphisms on some appropriate spaces and the key point is that for s small enough v has a norm < 1, which allows us to repeat the process convergently. Let us point out the fact that in general no common s can be chosen when the order of P increases indefinitely.

Due to this division theorem, we can build a canonical decomposition $\mathcal{E}$ of $\mathcal{U}$ -the standard fan- into convex polyhedral cones such that on each cone $\mathrm{gr}^L(h(I))$ is constant, where $h(I)$ is the homogenized ideal associated with I. Technically we prove first that the number of stairs $\mathrm{Exp}_L(h(I))$ and $\mathrm{Exp}_L(I)$ of I associated with all $L \in \mathcal{U}$ and a given total well ordering on $\mathbf{N}^{2n}$, is finite. The graded ideals $\mathrm{gr}^L(I)$ are also constant on each cone of $\mathcal{E}$, and the partition $\mathcal{E}$ finally appears as being the decomposition into the sets where $\mathrm{gr}^L(h(I))$ is fixed.

In the last chapter we give a geometric description of the cones in terms of the properties of $\mathrm{gr}^L(h(I))$. We prove that in each of the 4^n stratas, the relatively open cones are exactly those on which $\mathrm{gr}^L(h(I))$ is multihomogeneous of the appropriate type. We obtain a similar result concerning the restriction of the standard fan to any linear subspace and this implies that each cone in $\mathcal{E}$ is open in the affine space which it generates.

Finally, let us remark that the same kind of construction can be performed, with only extra notational difficulties for a submodule $\mathcal{N}$ of a free module $\mathcal{D}^r$.

Notations

Let $\mathcal{D}$ denote the $\mathbf{C}$-algebra of linear differential operators with coefficients in the ring $\mathbf{C}\{x_1, \ldots, x_n\} = \mathbf{C}\{x\}$ of convergent power series in n variables with complex coefficients, i.e. $\mathcal{D} = \mathbf{C}\{x\}[D_1, \ldots, D_n]$ with relations

$$[D_i, D_j] = 0, [D_i, a] = \frac{\partial a}{\partial x_i}$$

where $a \in \mathbf{C}\{x\}$.

Let $P = \sum_{\beta\in\mathbf{N}^n} p_\beta(x)D^\beta$ be a non-zero element of $\mathcal{D}$, where $p_\beta(x) \in \mathbf{C}\{x\}$. We denote by $\mathrm{ord}^F(P)$ the order of P (i.e. $\mathrm{ord}^F(P) = \max\{|\beta|; p_\beta(x) \neq 0\}$). We can write $P = \sum_{\alpha,\beta\in\mathbf{N}^n} p_{\alpha,\beta}x^\alpha D^\beta$ with $p_{\alpha,\beta} \in \mathbf{C}$ and we call the set $\mathcal{N}(P) = \{(\alpha, \beta) \in \mathbf{N}^{2n} \,|\, p_{\alpha,\beta} \neq 0\}$ the Newton diagram of P.

2 Finiteness results for ideals in $\mathcal{D}$

Let $\mathcal{U}$ be the set of linear forms $L : \mathbf{R}^{2n} \to \mathbf{R}$, $L(\alpha, \beta) = \sum_{i=1}^n e_i\alpha_i + \sum_{i=1}^n f_i\beta_i$ with $e_i + f_i \geq 0$ and $e_i \leq 0$ for $i = 1, \ldots, n$. Such a linear form L defines a filtration $F_{L,\bullet}$ (called the L-filtration) on $\mathcal{D}$ in the usual way. Let $P = \sum_{\beta\in\mathbf{N}^n} p_{\alpha,\beta}x^\alpha D^\beta$ be an element of $\mathcal{D}$. We define the L-order of P (denoted by $\mathrm{ord}^L(P)$) as the maximal value of $L(\alpha, \beta)$ over $\mathcal{N}(P)$. For each $k \in \mathbf{R}$ we have, by definition,

$$F_{L,k} = F_{L,k}(\mathcal{D}) = \{P \in \mathcal{D} \,|\, \mathrm{ord}^L(P) \leq k\}.$$

We denote by $\mathrm{gr}^L(\mathcal{D})$ the graded ring $\bigoplus_{k\in L(\mathbf{Z}^{2n})} \frac{F_{L,k}}{F_{L,<k}}$ associated with this filtration.

The structure of this ring is the following: after reordering if necessary the variables we may assume that

- $e_i < 0$ for $1 \leq i \leq p_2$ with $e_i + f_i = 0$ for $1 \leq i \leq p_1$ and $e_i + f_i > 0$ for $p_1 < i \leq p_2$
- $e_i = 0, f_i > 0$ for $p_2 < i \leq p_3$
- $e_i = f_i = 0$ for $p_3 < i \leq n$.

then $\mathrm{gr}^L(\mathcal{D})$ is isomorphic to

$$\mathbf{C}\{x_{p_2+1}, \ldots, x_n\}[x_1, \ldots, x_{p_2}, \xi_{p_1+1}, \ldots, \xi_{p_3}, D_1, \ldots, D_{p_1}, D_{p_3+1}, \ldots, D_n]$$

There is a finite number of types of these rings, one for each partition of $\{1, \ldots, n\}$ into four sets.

For each $P \in \mathcal{D}$ and for each $d \geq \mathrm{ord}^L(P)$ we denote by $\sigma_d^L(P)$ the symbol of order d of P with respect to the L-filtration, i.e. $\sigma^L(P)$ is the class of P in $F_{L,d}/F_{L,<d}$. We write $\sigma^L(P) = \sigma^L_{\mathrm{ord}^L(P)}(P)$ for the *principal symbol* of P. We have $\sigma^L(PQ) = \sigma^L(P)\sigma^L(Q)$ for $P, Q \in \mathcal{D}$.

For a left ideal I of $\mathcal{D}$ we denote by $\mathrm{gr}^L(I)$ the graded ideal associated with the induced L-filtration on I. This is the ideal of $\mathrm{gr}^L(\mathcal{D})$ generated by the set $\{\sigma^L(P) \,|\, P \in I\}$.

Theorem 2.1 *Let I be a left ideal of $\mathcal{D}$. Then the set $\{\mathrm{gr}^L(I) \,|\, L \in \mathcal{U}\}$ is finite.*

Let to this end $<$ be a total well ordering on $\mathbf{N}^{2n}$ compatible with sums and denote, for all $L \in \mathcal{U}$ the total ordering (but not in general a well ordering) on $\mathbf{N}^{2n}$ defined by:

$$(\alpha, \beta) <_L (\alpha', \beta') \quad \Leftrightarrow \quad \begin{cases} L(\alpha, \beta) < L(\alpha', \beta') \\ \text{or} \\ L(\alpha, \beta) = L(\alpha', \beta') \text{ and } |\beta| < |\beta'| \\ \text{or} \\ L(\alpha, \beta) = L(\alpha', \beta'), |\beta| = |\beta'| \text{ and } (\alpha', \beta') < (\alpha, \beta) \end{cases}$$

The total ordering $<_L$ is compatible with sums in $\mathbf{N}^{2n}$.

Definition 2.2 *For $P = \sum_{\alpha,\beta \in \mathbf{N}^n} p_{\alpha,\beta} x^\alpha D^\beta$ a non-zero element in $\mathcal{D}$ we define the L-privileged exponent of P as $\exp_L(P) = \max_{<_L}\{(\alpha, \beta) \in \mathbf{N}^{2n} \,|\, p_{\alpha,\beta} \neq 0\}$. Let I be a left ideal of $\mathcal{D}$. We denote by $\mathrm{Exp}_L(I)$ the set $\{\exp_L(P) \,|\, P \in I\}$.*

The L-privileged exponent of P is well defined because, for all $d \in \mathbf{N}$, the set $\{(\alpha, \beta) \,|\, |\beta| = d\}$ has a maximal element with respect to $<_L$. We have $\exp_L(PQ) = \exp_L(P) + \exp_L(Q)$ for $P, Q \in \mathcal{D}$.

Theorem 2.3 *Let I be a left ideal of $\mathcal{D}$. Then the set $\{\mathrm{Exp}_L(I) \,|\, L \in \mathcal{U}\}$ is finite.*

3 Homogenisation and finiteness results for homogeneous ideals in $\mathcal{D}[t]$

Let us denote by $\mathcal{D}[t]$ the algebra

$$\mathcal{D}[t] = \mathbf{C}\{x_1, \dots, x_n\}[t, D_1, \dots, D_n]$$

with relations

$$[t, a] = [t, D_i] = [a, b] = [D_i, D_j] = 0, [D_i, a] = \frac{\partial a}{\partial x_i} t$$

where $a, b \in \mathbf{C}\{x\}$. The algebra $\mathcal{D}[t]$ is a graded algebra; the graduation, in d, being

$$\mathcal{D}[t] = \bigoplus_{d \geq 0} \left(\bigoplus_{k + |\beta| = d} \mathbf{C}\{x\} t^k D^\beta \right).$$

Let $P = \sum_{\beta \in \mathbf{N}^n} p_\beta(x) D^\beta$ be an element of $\mathcal{D}$. The differential operator

$$h(P) = \sum_\beta p_\beta(x) t^{\mathrm{ord}^F(P) - |\beta|} D^\beta \in \mathcal{D}[t]$$

is called the homogenisation of P. If $H = \sum_{k,\beta} h_{k,\beta}(x) t^k D^\beta$ is an element of $\mathcal{D}[t]$, we denote by $H(1)$ the operator of $\mathcal{D}$

$$H(1) = \sum_{k,\beta} h_{k,\beta}(x) D^\beta.$$

With the notations above, for all $P, Q \in \mathcal{D}$ and for all homogeneous element $H \in \mathcal{D}[t]$,

1. $h(PQ) = h(P)h(Q)$.
2. There exists $k, l, m \in \mathbf{N}$ such that $t^k h(P + Q) = t^l h(P) + t^m h(Q)$.

3. There exists $k \in \mathbf{N}$ such that $t^k h(H(1)) = H$.

Let L be an element of $\mathcal{U}$. As in 2 such a linear form L defines a filtration $F_{L,\bullet}$ (called the L-filtration) on $\mathcal{D}[t]$ in the usual way. Let $G = \sum_{k,\beta\in\mathbf{N}^n} g_{k,\alpha,\beta} t^k x^\alpha D^\beta$ be an element of $\mathcal{D}[t]$. We define the L-order of G (denoted by $\mathrm{ord}^L(G)$) as the maximal value of $L(\alpha,\beta)$ over the set of (k,α,β) such that $g_{k,\alpha,\beta} \neq 0$. For each $k \in \mathbf{R}$ we have, by definition,

$$F_{L,k}(\mathcal{D}[t]) = \{G \in \mathcal{D}[t] \mid \mathrm{ord}^L(G) \leq k\}.$$

We denote by $\mathrm{gr}^L(\mathcal{D}[t])$ the graded ring associated with this filtration. We have $\mathrm{gr}^L(\mathcal{D}[t]) = \mathrm{gr}^L(\mathcal{D})[t]$ where, with the notations of 2, $[D_i, x_i] = t$ if $e_i + f_i = 0$ and $[\xi_i, x_i] = 0$ if $e_i + f_i > 0$.

For each $G \in \mathcal{D}[t]$ we denote by $\sigma^L(G)$ the principal symbol of G with respect to the L-filtration.

We denote by $\widehat{\mathcal{D}}$ the algebra $\mathbf{C}[[x]][D] = \mathbf{C}[[x_1,\ldots,x_n]][D_1,\ldots,D_n]$ of linear differential operators with formal power series coefficients. As before we denote by $\widehat{\mathcal{D}}[t]$ the corresponding graded algebra.

For a left homogeneous ideal J of $\mathcal{D}[t]$ (resp. $\widehat{\mathcal{D}}[t]$) we denote by $\mathrm{gr}^L(J)$ the graded ideal associated with the induced L-filtration on J. This ideal is generated by the set $\{\sigma^L(G) \mid G \in J\}$.

Theorem 3.1 *Let J be a left homogeneous ideal of $\mathcal{D}[t]$. Then the set $\{\mathrm{gr}^L(J) \mid L \in \mathcal{U}\}$ is finite. And the same is true for $\widehat{\mathcal{D}}[t]$.*

Let $<_L$ be the total ordering on $\mathbf{N}^{2n}$ defined in 2. We recall that the extension of $<_L$, denoted by $<_L^h$, is the total ordering on $\mathbf{N}^{1+2n}$ (compatible with sums) defined by:

$$(k,\alpha,\beta) <_L^h (k',\alpha',\beta') \iff \begin{cases} k + |\beta| < k' + |\beta'| \\ \text{or} \begin{cases} k + |\beta| = k' + |\beta'| \text{ and} \\ (\alpha,\beta) <_L (\alpha',\beta') \end{cases} \end{cases}$$

Since $<_L^h$ is a total ordering (but not a well ordering) compatible with sums, we have for all non-zero element

$$G = \sum_{a,\alpha,\beta} g_{(a,\alpha,\beta)} t^a \underline{x}^\alpha \underline{D}^\beta \in \mathcal{D}[t]$$

the notion of privileged exponent of G w.r.t. $<_L^h$, which we denote by $\exp_{<_L^h}(G)$ (or simply $\exp_L(G)$): If $\mathcal{N}(G) = \{(a,\alpha,\beta); g_{(a,\alpha,\beta)} \neq 0\}$ denotes the Newton diagram of G, then $\exp_L(G) = \max_{<_L^h} \mathcal{N}(G)$. Also we have for all non-zero ideal J of $\mathcal{D}[t]$, the notion of a Gröbner (or standard) basis of J. We denote by

$$\mathrm{Exp}_L(J) = \mathrm{Exp}_{<_L^h}(J) = \{\exp_{<_L^h}(P) \mid P \in J\}.$$

Theorem 3.2 *Let J be a left homogeneous ideal of $\mathcal{D}[t]$. Then the set $\{\mathrm{Exp}_L(J) \mid L \in \mathcal{U}\}$ is finite. And the same is true for $\widehat{\mathcal{D}}[t]$.*

Theorem 3.2 implies theorem 3.1. The proofs are given in 5.

Let $\pi : \mathbf{N}^{1+2n} = \mathbf{N} \times \mathbf{N}^{2n} \to \mathbf{N}^{2n}$ denote the natural projection. The proofs of 1,2,3 and 4 below are elementary and left to the reader:

1. If $P \in \mathcal{D}$, then $\pi(\exp_{<_L^h}(h(P))) = \exp_{<_L}(P)$.

2. More generally, if H is an homogeneous element of $\mathcal{D}[t]$, then

$$\pi(\exp_{<_L^h}(H)) = \pi(\exp_{<_L^h}(h(H(1)))) = \exp_{<_L}(H(1)).$$

Let I be a left ideal of $\mathcal{D}$. We denote by $h(I)$ the homogeneous ideal of $\mathcal{D}[t]$, generated by $\{h(P) \mid P \in I\}$. We call $h(I)$ the homogenized ideal of I. With these notations we have the following

3. $\pi(\mathrm{Exp}_{<_L^h}(h(I))) = \mathrm{Exp}_{<_L}(I)$.

4. Let $\{P_1, \dots, P_m\}$ be a generating system of I and let $\widetilde{I}$ be the ideal generated by $\{h(P_1), \dots, h(P_m)\}$ in $\mathcal{D}[t]$. Then $\pi(\mathrm{Exp}_{<^h_L}(\widetilde{I})) = \mathrm{Exp}_{<_L}(I)$.

Let $F_\bullet(\mathcal{D})$ denote the filtration on $\mathcal{D}$ by the order of the differential operators (see 1). If P is a differential operator in $\mathcal{D}$, then we denote by $\sigma^F(P)$ the principal symbol of P w.r.t. this filtration. If I is an ideal of $\mathcal{D}$, then we denote by $\mathrm{gr}^F(I)$ the graded ideal associated with the induced filtration on I.

Lemma 3.3 *Let I be a left ideal of $\mathcal{D}$ and let $\{P_1, \dots, P_m\}$ be a family of differential operators of I. The following assertions are equivalent:*

1. $h(I) = (h(P_1), \dots, h(P_m))$.
2. $\mathrm{gr}^F(I) = (\sigma^F(P_1), \dots, \sigma^F(P_m))$.

4 Division Theorem in $\mathcal{D}[t]$

Let $L : \mathbf{R}^{2n} \to \mathbf{R}$ be a linear form, $L(\alpha, \beta) = \sum_{i=1}^n e_i\alpha_i + \sum_{i=1}^n f_i\beta_i$ with $e_i + f_i \geq 0$ and $e_i \leq 0$ for all i. Let $\{P_1, \dots, P_r\}$ be a family of non-zero homogeneous elements of $\mathcal{D}[t]$. Write $d_j = \mathrm{ord}^T(P_j)$ (where ord^T denote the order with respect to (t, D)) and $\delta_j = \mathrm{ord}^L(P_j)$.

Denote by $(\Delta_1, \dots, \Delta_r, \overline{\Delta})$ the partition of $\mathbf{N}^{1+2n}$ defined by $(\exp_L(P_1), \dots, \exp_L(P_r))$. We have:

- $\Delta_1 = \exp_L(P_1) + \mathbf{N}^{1+2n}$.
- $\Delta_i = (\exp_L(P_i) + \mathbf{N}^{1+2n}) \setminus (\cup_{j=1}^{i-1}\Delta_j)$, for $i = 2, \dots, r$.
- $\overline{\Delta} = \mathbf{N}^{1+2n} \setminus (\cup_{j=1}^{r}\Delta_j)$

Theorem 4.1 *For any $P \in \mathcal{D}[t]$ there exists $(Q_1, \dots, Q_r, R) \in (\mathcal{D}[t])^{r+1}$ such that:*

1. $P = \sum_i Q_iP_i + R$
2. *If $Q_i \neq 0$ then $\mathcal{N}(Q_i) + \exp_L(P_i) \subset \Delta_i$ for all i*
3. *If $R \neq 0$ then $\mathcal{N}(R) \subset \overline{\Delta}$*

Proposition 4.2 *Let $P_1, \dots, P_r$ be homogeneous elements of $\mathcal{D}_n[t]$, and L a linear form in $\mathcal{U}$. Then there exist linear forms L' in $\mathcal{U}$, arbitrarily near to L and with only non zero coefficients such that $e_j + f_j > 0$ for any j and $\exp_{L'}(P_i) = \exp_L(P_i)$ for any i. Furthermore we may assume that $\sigma_{L'}(P_i)$ is a monomial for all i and L' has rational coefficients.*

Proof *of the proposition.* Let us assume to simplify the notation that for some p we can write $L(\alpha, \beta) = \sum_{j=1}^p \alpha_je_j + \sum_{j=1}^n \beta_jf_j$, with $e_1 > 0, \dots, e_p > 0$ and of course $e_{p+1} = 0, \dots, e_n = 0$.

Let us denote by $\mathcal{N}(P_i)$ the Newton diagram of P_i. Consider the set

$$F_i = \bigcup_{(l,\alpha,\beta)\in\mathcal{N}(P_i)} (l - \mathbf{N}) \times (\alpha + \mathbf{N}^n) \times (\beta - \mathbf{N}^n).$$

Of course F_i is invariant by the translations of $(-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)$ and it is finitely generated (over $(-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)$) since P_i is "polynomial" in (t, D). So we can write

$$F_i = \bigcup_{k=1}^{s_i} ((l^{i,k}, \alpha^{i,k}, \beta^{i,k}) + (-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)).$$

We denote by $k_i \in \{1, \cdots, s_i\}$ the index such that $(l^{i,k_i}, \alpha^{i,k_i}, \beta^{i,k_i}) = \exp_L(P_i)$. For any k we have $L(\alpha^{i,k_i}, \beta^{i,k_i}) \geq L(\alpha^{i,k}, \beta^{i,k})$ and for any L' and any i the privileged exponent $\exp_{L'}(P_i)$ has to be chosen in the finite set of all $(l^{i,k}, \alpha^{i,k}, \beta^{i,k})$. By definition of $\exp_L(P_i)$, we have for any $k \in \{1, \cdots, s_i\}$:

$$L(\alpha^{i,k_i}, \beta^{i,k_i}) - L(\alpha^{i,k}, \beta^{i,k}) = \sum_{j=1}^{p} (\alpha_j^{i,k_i} - \alpha_j^{i,k}) e_j + \sum_{j=1}^{n} (\beta_j^{i,k_i} - \beta_j^{i,k}) f_j \geq 0$$

Consider the form $L_\epsilon(\alpha, \beta) = \sum_{j=1}^{p} \alpha_j e_j + \sum_{j=1}^{n} \beta_j (f_j + \epsilon)$. Then we get:

$$L_\epsilon(\alpha^{i,k_i}, \beta^{i,k_i}) - L_\epsilon(\alpha^{i,k}, \beta^{i,k}) = L(\alpha^{i,k_i}, \beta^{i,k_i}) - L(\alpha^{i,k}, \beta^{i,k}) + \epsilon(|\beta^{i,k_i}| - |\beta^{i,k}|)$$

The definition of the order $<_L^h$ implies then that $L_\epsilon(\alpha^{i,k_i}, \beta^{i,k_i}) - L_\epsilon(\alpha^{i,k}, \beta^{i,k}) \geq 0$ for any k.
Furthermore when $L(\alpha^{i,k_i}, \beta^{i,k_i}) = L(\alpha^{i,k}, \beta^{i,k})$ we have the inequalities

$$(\alpha^{i,k_i}, \beta^{i,k_i}) < (\alpha^{i,k}, \beta^{i,k})$$

and it is well known (see for example [4]) that for a finite number of multi-indices the restriction of any well ordering can be described by a linear form Λ with strictly positive coefficients: Thus if $L(\alpha^{i,k_i}, \beta^{i,k_i}) = L(\alpha^{i,k}, \beta^{i,k})$, $\Lambda(\alpha^{i,k_i}, \beta^{i,k_i}) - \Lambda(\alpha^{i,k}, \beta^{i,k}) \leq 0$. If we set $L' = L_\epsilon - \eta\Lambda$ we obtain a linear form which for ϵ small enough and then with $\eta > 0$ small enough with respect to ϵ has only positive coefficients and satisfies the condition in the proposition. The assertion about the rationality of the coefficients of L' comes from the openness of the conditions in the statement of the proposition. □

Proof *of the theorem 4.1.* The results of the theorem depend only on the $\exp_L(P_i)$ and then if they are true for L' in proposition 4.2 they are also true for L. Therefore we may assume that L is a strict rational linear form (i.e. e_i, f_i are rational, $e_i < 0$, $f_i > 0$, $e_i + f_i > 0$) and furthermore we may assume that for all j

$$\sigma^L(P_j) = t^{k_j} x^{\alpha_j} D^{\beta_j}.$$

By the rationality of L there exists $h \in \mathbf{N}$, $h > 0$ such that $hL(\mathbf{Z}^{2n}) \subset \mathbf{Z}$.

We fix $d \in \mathbf{N}$. We denote by E_d the complex vector space of homogeneous operators in $\mathcal{D}[t]$ of total degree equal to d.

Let $P = \sum_{k,\alpha,\beta} p_{k,\alpha,\beta} t^k x^\alpha D^\beta$ be an operator in E_d. We consider the pseudo-norm on E_d defined by $|P|_s = \sum |p_{k,\alpha,\beta}| s^{-L(\alpha,\beta)}$ where $s \in \mathbf{R}_+^*$. The following lemma will be used later and its proof is left to the reader.

Lemma 4.3 *Let $P \in \mathcal{D}[t]$ and $s_0 > 0$. Then for all $0 < s < s_0$ and for all $\delta \geq \mathrm{ord}^L(P)$ we have $|P|_s \leq (\frac{s}{s_0})^{-\delta} |P|_{s_0}$.*

We consider the family of Banach spaces

$$E_{d,s} = \{P \in E_d \,|\, |P|_s < +\infty\}.$$

We have $E_{d,s} \subset E_{d,s'}$ for $s' \leq s$ (because $|P|_s < +\infty$ iff $\sum_{L(\alpha,\beta)>0} |p_{k,\alpha,\beta}| s^{-L(\alpha,\beta)} < +\infty$) and $E_d = \bigcup_{s>0} E_{d,s}$.

Lemma 4.4 *Let $\widetilde{P}$ be an L-homogeneous element of E_d with $\mathrm{ord}^L(\widetilde{P} = \delta$. The there exist $\widetilde{Q_j} \in E_{d-d_j}$ for $j = 1, \ldots, r$, $\widetilde{R} \in E_d$ and $\widetilde{S} \in E_d$ such that:*

1. *$\widetilde{Q_j}$ is L-homogeneous and $\mathrm{ord}^L(\widetilde{Q_j}) = \delta - L(\alpha_j, \beta_j) = \delta - \delta_j$.*
2. *$\widetilde{R}$ is L-homogeneous and $\mathrm{ord}^L(\widetilde{R}) = \delta$.*
3. *$\mathrm{ord}^L(\widetilde{S}) \leq \delta - 1/h$.*
4. *$\widetilde{P} = \sum_j \widetilde{Q_j} P_j + \widetilde{R} + \widetilde{S}$.*

5. $\mathcal{N}(\widetilde{Q_j}) + \exp_L(P_j) \subset \Delta_j$ *and* $\mathcal{N}(\widetilde{R}) \subset \overline{\Delta}$.

Moreover, there exists $K > 0$ *and* $s_0 > 0$ *such that for all* $0 < s \le s_0$ *we have* $|\widetilde{Q_j}|_s \le s^{-\delta_j}|\widetilde{P}|_s, |\widetilde{R}|_s \le |\widetilde{P}|_s$, $|\widetilde{S}|_s \le Ks^{1/h}|\widetilde{P}|_s$ *and* K *depends only on* $P_1, \dots, P_r$ *and* d.

Remark 4.5 *The norm* $|\widetilde{P}|_s$ *is equal to* $C_{\widetilde{P}}s^{-\delta}$ *for some positive constant* $C_{\widetilde{P}}$.

Proof *(of the Lemma)* It is enough to work with t-homogeneous operators in E_d. So we can suppose

$$\widetilde{P} = t^k A(x, D)$$

where $A(x, D) \in \mathcal{D}$ is D-homogeneous. By commutative division in the ring $\mathbf{C}[t, x, \xi]$ we have

$$t^k A(x, \xi) = \sum_{j=1}^{r} t^{k-k_j} A_j(x, \xi) t^{k_j} x^{\alpha_j} \xi^{\beta_j} + t^k R(\widetilde{P})(x, \xi)$$

where $A_j(x, \xi)$ is L-homogeneous with $\mathrm{ord}^L(A_j) = \delta - L(\alpha_j, \beta_j)$ ($A_j = 0$ if $k < k_j$) and $R(\widetilde{P})$ is L-homogeneous with $\mathrm{ord}^L(R(\widetilde{P})) = \delta$. Remark that $\exp_L(P_j) = (k_j, \alpha_j, \beta_j)$ and so $(k_j, \alpha_j, \beta_j) + \mathcal{N}(t^{k-k_j}A_j(x, \xi)) \subset \Delta_j$ and $\mathcal{N}(t^k R(\widetilde{P})) \subset \overline{\Delta}$.

We now consider the operator in $\mathcal{D}[t]$ defined by the product

$$A_j(x, D) x^{\alpha_j} D^{\beta^j} = \sum a_{j,\alpha,\beta} x^\alpha D^\beta x^{\alpha_j} D^{\beta_j}.$$

Then we obtain

$$\widetilde{P} = \sum_{j=1}^{r} t^{k-k_j} A_j(x, D) t^{k_j} x^{\alpha_j} D^{\beta_j} + t^k R(\widetilde{P})(x, D) + W_1(t, x, D)$$

for some operator W_1 in $\mathcal{D}[t]$ and then

$$\widetilde{P} = \sum_{j=1}^{r} t^{k-k_j} A_j(x, D) P_j + t^k R(\widetilde{P})(x, D) + W_2(t, x, D)$$

where $W_2 = W_1 + \sum_j t^{k-k_j} A_j(x, D)(t^{k_j} x^{\alpha_j} D^{\beta_j} - P_j)$ is of L-order strictly less than δ.

- We have $|A|_s \le |\widetilde{P}|_s$ and $|B|_s \le |\widetilde{P}|_s$. Then

$$\sum_j |A_j|_s |t^{k_j} x^{\alpha_j} D^{\beta_j}|_s = \sum_j s^{-\delta_j)} |A_j|_s \le |A|_s \le |\widetilde{P}|_s$$

  and $|R(\widetilde{P})|_s \le |A|_s \le |\widetilde{P}|_s$.

- $W_1 = \sum_{j,\alpha,\beta} t^k a_{j,\alpha,\beta} x^\alpha [D^\beta, x^{\alpha_j}] D^{\beta_j}$. We have

$$|W_1|_s \le \sum_{j,\alpha,\beta} |a_{j,\alpha,\beta}| s^{-L(\alpha,0)} \left(\sum_{\gamma \ne 0} C_{j,\beta,\gamma} s^{-L(\alpha_j - \gamma, \beta + \beta_j - \gamma)} \right) =$$

$$= \sum_{j,\alpha,\beta} |a_{j,\alpha,\beta}| s^{-L(\alpha,\beta)} u_{j,\beta}(s)$$

  where $C_{j,\beta,\gamma}$ is some constant and $u_{j,\beta}(s) = \sum_{\gamma \ne 0} C_{j,\beta,\gamma} s^{-L(\alpha_j - \gamma, \beta_j - \gamma)} = s^{-\delta_j} \sum_{\gamma \ne 0} C_{j,\beta,\gamma} s^{L(\gamma,\gamma)}$. Because $L(\gamma, \gamma) > 0$ there exists $K_s > 0$ (increasing function of s) such that $u_{j,\beta}(s) \le K_s s^{-\delta_j + 1/h}$ (remember that h is a positive integer verifying $hL(\mathbf{Z}^{2n}) \subset \mathbf{Z}$). Finally we have, for all s,

$$|W_1|_s \le K_s s^{1/h} \sum |a_{j,\alpha,\beta}| s^{-L(\alpha,\beta) - \delta_j} = K_s s^{1/h} \sum |A_j|_s s^{-\delta_j} \le K_s s^{1/h} |\widetilde{P}|_s.$$

- Write $P_j = t^{k_j} x^{\alpha_j} D^{\beta_j} + \sum_{j,k',\beta'} p_{j,k',\beta'}(x) t^{k'} D^{\beta'}$. We have $\text{ord}^L(p_{j,k',\beta'}(x) D^{\beta'} < \delta_j$. As in the preceding point (making, perhaps, K_s bigger and by applying lemma 4.3 to $\partial^\gamma(p_{j,k',\beta'})$ when $\gamma_i \le \beta_i$ for all i) we have $|W_2|_s \le K s^{1/h} |\widetilde{P}|_s$ for all $s \le s_0$; where s_0 is chosen in order that $|p_{j,k',\beta'}|_{s_0} < +\infty$ for all j, k', β' and $K = K_{s_0}$.

Then, we define $\widetilde{Q_j} = t^{k-k_j} A_j(x, D)$, $\widetilde{R} = t^k R(\widetilde{P})$ and $\widetilde{S} = W_2$ and the lemma is proved. □

Now we return to the proof of the theorem. Consider, for $0 < s \le s_0$ $E_{d,s,\delta} = \{P \in E_{d,s} \,|\, P \text{ is } L-\text{homogeneous of } L-\text{order } \delta\}$ and $E_{d,s,\le\delta} = \{P \in E_{d,s} \,|\, \text{ord}^L(P) \le \delta\}$.

By lemma we have continuous linear maps

- $q_{j,\delta} : E_{d,s,\delta} \to E_{d,s,\delta-\delta_j}$ with $q_{j,\delta}(\widetilde{P}) = \widetilde{Q_j}$.
- $r_\delta : E_{d,s,\delta} \to E_{d,s,\delta}$ with $r_\delta(\widetilde{P}) = \widetilde{R}$.
- $v_\delta : E_{d,s,\delta} \to E_{d,s,\le\delta-1/h}$ with $v_\delta(\widetilde{P}) = \widetilde{S}$.

Let $P = \sum_k P_{\delta-k/h}$ be an element of $E_{d,s}$, where $\delta = \text{ord}^L(P)$ and $P_{\delta-k/h}$ is L-homogeneous of ord^L equal to $\delta - k/h$. The series

$$\sum_k v_{\delta-k/h}(P_{\delta-k/h})$$

is absolutely convergent in $E_{d,s}$ (because $|v_{\delta-k/h}(P_{\delta-k/h})|_s \le K s^{1/h} |P_{\delta-k/h}|_s)$.
So we define a linear map $v : E_{d,s} \to E_{d,s}$ by $v(P) = \sum_k v_{\delta-k/h}(P_{\delta-k/h})$. We also have linear maps $q_j : E_{d,s} \to E_{d,s}$ by $q_j(P) = \sum_k q_{j,\delta-k/h}(P_{\delta-k/h})$ and $r : E_{d,s} \to E_{d,s}$ by $r(P) = \sum_k r_{\delta-k/h}(P_{\delta-k/h})$.

We have:

- $|q_j(P)|_s \le \sum_k |q_{j,\delta-k/h}(P_{\delta-k/h})|_s \le \sum_k s^{-\delta_j} |P_{\delta-k/h}|_s = s^{-\delta_j} |P|_s$.
- $|r(P)|_s \le |P|_s$.
- $|v(P)|_s \le K s^{1/h} |P|_s$.
- $P = \sum_j q_j(P) P_j + r(P) + v(P)$.

We fix $s_1 > 0$ such that $s_1 \le s_0$ and $\epsilon = K s_1^{1/h} < 1$. So, writing $v^0(P) = P$ and $v^l(P) = v(v^{l-1}(P))$, the series

$$\sum_l v^l(P)$$

converges in E_{d,s_1} because $|v^l(P)|_s \le \epsilon^l |P|_s$. The same is true for $Q_j = \sum_l q_j(v^l(P))$ and $R = \sum_l r(v^l(P))$. Finally taking limits we have

$$P = \sum_j Q_j P_j + R$$

where Q_j and R verify the conditions in the statement. □

Remarks 4.6 *For given d and s there is a common polydisc of convergence for the remainder of the division and we have obtained in the proof of the theorem 4.1 a continuous division $E_{d,s_1} \to (E_{d,s_1})^{r+1}$.*

This is not the case independently of d. Consider for example in the one variable case the homogeneous operator $P = t\partial - x^2\partial^2$ and the division by P of the degree $d+1$ monomials $M_p = t^p \partial^{d+1-p}$ for $p = 1, \dots, d$. The first step of the division gives:

$$M_p = t^{p-1}\partial^{d-p}(P + x^2\partial^2) = t^{p-1}\partial^{d-p}P + x^2 M_{p-1} + 2(d-p)xM_p + 2(d-p)(d-p-1)M_{p+1}.$$

By an argument of linear algebra we can find $M \in E_{d+1}$ (M polynomial in x) and $\lambda \in \mathbf{C}$ with $|\lambda| \ge d$ such that $M = QP + R + \lambda x M$.

Thus the division process of M by P generates a remainder with radius of convergence $\rho \le (1/|\lambda|) \le (1/d)$.

Corollary 4.7 *We have* $\exp_L(P) = \max\{\exp_L(R), \exp_L(Q_i P_i),\ i = 1, \dots, r\}$ *and* $\sigma^L(P) = \sum \sigma^L_{\delta-\delta_i}(Q_i)\sigma^L(P_i) + \sigma^L_\delta(R)$ *where* $\delta = \mathrm{ord}^L(P), \delta_i = \mathrm{ord}^L(P_i)$.

Remark 4.8 *We have a division theorem in* $\widehat{\mathcal{D}}[t]$. *Its statement is analogous to the one in theorem 4.1 and its proof is simpler.*

Notation 4.9 *The element* R *in the theorem above is denoted by* $r(P; P_1, \dots, P_r)$.

Corollary 4.10 *A family* $\{P_1, \dots, P_m\}$ *of elements of an ideal* K *of* $\mathcal{D}[t]$ *(or* $\widehat{\mathcal{D}}[t]$*) is a* L*-standard basis of* K *if and only if* $r(P; P_1, \dots, P_m) = 0$ *for all* $P \in K$.

Let P_1, P_2 be two operators with privileged exponents $\lambda_1 = (k_1, \alpha^1, \beta^1), \lambda_2 = (k_2, \alpha^2, \beta^2)$. We call the semi-syzygy of P_1, P_2 the operator $M_1P_1 - M_2P_2 = S(P_1, P_2)$ where M_1, M_2 are two monomials whose exponents ν^1, ν^2 are such that $\nu^1 + \lambda_1 = \nu^2 + \lambda_2$ and minimal for this property and furthermore such that the leading coefficients of M_1P_1 and M_2P_2 are equal so that we get $\exp_L(S(P_1, P_2)) <^h_L \exp_L(M_1P_1) = \exp_L(M_2P_2)$.

Proposition 4.11 *Let* $F = \{P_1, \dots, P_r\}$ *be a system of generators of an homogeneous ideal* J *of* $\mathcal{D}[t]$. *The following are equivalent:*

1. $\mathcal{F}$ *is a* L*-standard basis of* J
2. $r(S(P_i, P_j); P_1, \dots, P_r) = 0$ *for any* (i, j).

Proof. Because of the corollary 4.10 it is enough to prove 2) $\Rightarrow$ 1). So, we must prove the equality

$$\mathrm{Exp}_L(J) = \bigcup_{i=1}^{r} (\exp_L(P_i) + \mathbf{N}^{1+2n})$$

knowing that the rest of the divisions of any $S(P_i, P_j)$ is zero.
We suppose first that the linear form L is strict and rational (as in the proof of theorem 4.1).

The proof follows the same line as in [6], [15] except that we must be careful about the questions of finiteness.

Let P be a non zero element of J. For any decomposition of P on the generators $\{P_i\}$:

$$P = \sum_{i=1}^{r} H_i P_i,$$

we define :

$$\underline{H} = (H_1, \dots, H_r)$$
$$\delta(\underline{H}) = \max_i \{\exp_L(H_i P_i)\}$$
$$L(\underline{H}) = \max_i \{\mathrm{ord}^L(H_i P_i\}.$$

By the usual process involving the division of the $S(P_i, P_j)$ we can replace $\underline{H}$ by $\underline{H'}$ such that $P = \sum_i H'_i P_i$ and $\delta(\underline{H'}) < \delta(\underline{H})$. Since the division is homogeneous with respect to the variables (t, D), $|\beta|$ is bounded in the developments of the H_i, H'_i, and for a fixed value of $L(\underline{H})$ there are only a finite number of possible $\delta(\underline{H})$, because L is strict. Therefore $L(\underline{H})$ decreases after repeating the first process a finite number of times.

Finally since the set $L(\mathbf{Z}^{2n})$ is discrete (or again because $|\beta|$ is bounded) we can find $\underline{H''}$ with $P = \sum_i H''{}_i P_i$ and $\exp_L(P) = \delta(\underline{H''}) \in \cup_i (\exp_L(P_i) + \mathbf{N}^{1+2n})$.

Suppose now that L is general and let P be an element of J. By 4.2, there exists a strict linear form L' such that $\exp_L(P_i) = \exp_{L'}(P_i)$ and $\exp_L(P) = \exp_{L'}(P)$. By the first part of the proof $\{P_1, \dots, P_r\}$ is a L'-standard basis of J and therefore;

$$\exp_L(P) = \exp_{L'}(P) \in \bigcup_{i=1}^{r} (\exp_{L'}(P_i) + \mathbf{N}^{1+2n}) = \bigcup_{i=1}^{r} (\exp_L(P_i) + \mathbf{N}^{1+2n})$$

□

5 Proofs

We will start by a key technical result which is in fact a particular case of theorem 3.1 and 3.2.

Proposition 5.1 *Let P be a non-zero element of $\mathcal{D}[t]$. The sets $\{\exp_L(P) \mid L \in \mathcal{U}\}$ and $\{\sigma^L(P) \mid L \in \mathcal{U}\}$ are finite. The same is true in $\widehat{\mathcal{D}}[t]$.*

Proof. Let us denote by $\mathcal{N}(P)$ the Newton diagram of P. Consider the set

$$F = \bigcup_{(l,\alpha,\beta)\in\mathcal{N}(P)} (l - \mathbf{N}) \times (\alpha + \mathbf{N}^n) \times (\beta - \mathbf{N}^n).$$

Of course F is invariant by the translations of $(-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)$ and it is finitely generated (over $(-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)$) since P is "polynomial" in (t, D). So we can write

$$F = \bigcup_{k=1}^{s} ((l^k, \alpha^k, \beta^k) + (-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)).$$

Suppose that for some $L \in \mathcal{U}$ and some $(l, \alpha, \beta) \in \mathbf{N}^{1+2n}$ we have $\exp_L(P) = (l, \alpha, \beta)$: there exists k such that

$$(l, \alpha, \beta) = (l^k, \alpha^k, \beta^k) + (l', \gamma, -\gamma')$$

for some $l' \in \mathbf{N}$ and $\gamma, \gamma' \in \mathbf{N}^n$. So we have $L(\alpha, \beta) \le L(\alpha^k, \beta^k)$ and then, by definition of $\exp_L(P)$, $L(\alpha, \beta) = L(\alpha^k, \beta^k)$. Thus, there exists only a finite number of possible values for $\exp_L(P)$ when L varies in $\mathcal{U}$.

Therefore we have $\sigma^L(P) = \sum_{L(\alpha,\beta)=\mathrm{ord}^L(P)} a_{l,\alpha,\beta} t^l x^\alpha D^\beta$ and the exponents (α, β) for which $L(\alpha, \beta) = \mathrm{ord}^L(P)$ are determined by the following rule:

- $(l, \alpha, \beta) \in \mathcal{N}(P)$
- There exists k such that $(l, \alpha, \beta) \in (l^k, \alpha^k, \beta^k) + (-\mathbf{N}) \times \mathbf{N}^n \times (-\mathbf{N}^n)$
- $(\alpha_i - \alpha_i^k)e_i = 0$ and $(\beta_i - \beta_i^k)f_i = 0$ for any i.

For a fixed P there is only a finite number of choices for $\sigma^L(P)$ following this rule. □

5.1 Homogeneous results

Proof of theorem 3.2. Suppose the result were not true. Then there would exist an infinite set of linear forms in $\mathcal{U}$, $\mathcal{L} = \{L_i\}_{i\ge1}$ such that, writing $E_i = \mathrm{Exp}_{L_i}(J)$, we have $E_i \ne E_j$ for $i \ne j$. Let $\{P_1, \dots, P_m\}$ be a system of generators of the ideal J. Thanks to the technical result 5.1 we can suppose that, for each l we have $\exp_{L_i}(P_l) = \exp_{L_j}(P_l)$ for all i, j. Consider the set

$$W_0 = \bigcup_{l=1}^{m} (\exp_{L_i}(P_l) + \mathbf{N}^{1+2n})$$

which is in fact independent of i. We can suppose that $W_0 \ne E_i$ for all i (in fact this inequality is true for all i except maybe one). So the set $\{P_1, \dots, P_m\}$ is not a standard basis of J w.r.t all $<_{L_i}$. Then there exists $P_{m+1} \in J$ such that $\exp_{L_1}(P_{m+1}) \notin W_0$. By means of the formal division theorem we can assume that $\mathcal{N}(P_{m+1}) \cap W_0 = \emptyset$. Now, by 5.1 we can also assume that $\exp_{L_i}(P_{m+1}) = \exp_{L_j}(P_{m+1})$ for all i, j. We restart with $\{P_1, \dots, P_{m+1}\}$ as a system of generators of J and with $W_1 = \bigcup_{l=1}^{m+1} (\exp_{L_i}(P_l) + \mathbf{N}^{1+2n})$. This produces an infinite chain $W_0 \subsetneq W_1 \subsetneq \cdots$ which is impossible because the noetherianity of $\mathbf{N}^{1+2n}$. □

Proof of theorem 3.1 We deduce 3.1 from 3.2 in the following way. Let E be a subset of $\mathbf{N}^{1+2n}$ invariant by the translations of $\mathbf{N}^{1+2n}$ and let L be a linear form such that $\mathrm{Exp}_L(J) = E$. We have to prove that the set of all $\mathrm{gr}^{L'}(J)$ such that $\mathrm{Exp}_{L'}(J) = E$ is finite. By the division theorem we can find $\mathcal{B} = \{P_1, \dots, P_m\}$ with the two following properties:

- $\mathcal{B}$ is an L-standard basis of J.
- For all i, $(\mathcal{N}(P_i) \setminus \{\exp_L(P_i)\}) \subset (\mathbf{N}^{1+2n} \setminus E)$.

These conditions imply that $\mathcal{B}$ is an L'-standard basis of J for any L' such that $\mathrm{Exp}_{L'}(J) = E$. ¿From this we deduce that

$$\mathrm{gr}^{L'}(J) = (\sigma^{L'}(P_1), \dots, \sigma^{L'}(P_m)).$$

To conclude the proof we have only to apply the analogue of 5.1 to the rings $\mathrm{gr}^{L'}(\mathcal{D}[t])$ and the elements $\sigma^{L'}(P_i)$ of these rings. According to 2 recall that these rings are canonically isomorphic to a finite number of them and that the finiteness of the number of $\sigma^{L'}(P_i)$ must be understood via these isomorphisms. □

Following the above proof we can state the following definition and proposition:

Definition 5.2 *The set $\mathcal{B} = \{Q_1, \dots, Q_r\}$ is a reduced standard basis of the homogeneous ideal J if*

- *$\mathcal{B}$ is an L-standard basis of J.*
- *For all i, $\mathcal{N}(Q_i) \setminus \{\exp_L(Q_i)\} \subset \mathbf{N}^{1+2n} \setminus E$*

Proposition 5.3 *Any homogeneous ideal admits a unique reduced L-standard basis made of homogeneous operators. Let $\mathcal{B}$ be the L-reduced standard basis of J. For any L' such that $\exp_L(Q_i) = \exp_{L'}(Q_i)$ then $\mathcal{B}$ is the L'-reduced standard basis.*

5.2 Finiteness results in $\mathcal{D}_n$

Proof of theorem 2.3 Let $J = h(I)$ be the ideal generated by all the homogeneous elements $h(P)$ such that $P \in I$. Let also $\pi : \mathbf{N}^{1+2n} \longrightarrow \mathbf{N}^{2n}$ be the natural projection $(k, \alpha, \beta) \mapsto (\alpha, \beta)$. We have (see 3) $\pi(\mathrm{Exp}_L(h(I))) = \mathrm{Exp}_L(I)$. Since the set of all $\mathrm{Exp}_L(h(I))$ is finite by 3.2, this proves the finiteness of the set of all $\mathrm{Exp}_L(I)$. □

Proof of theorem 2.1 In the theorem 2.1 the finiteness of the set of $\mathrm{gr}^L(I)$ must be understood in the following way: These are graded ideals of the pair-wise distinct graded rings $\mathrm{gr}^L(\mathcal{D}_n)$ which can be collected in a finite set of isomorphism classes, up to unique non graded isomorphisms.
The operation $t = 1$ gives a morphism of graded rings:

$$\Phi_L : gr^L(\mathcal{D}_n[t]) \to gr^L(\mathcal{D}_n)$$

and we deduce the theorem 2.1 from the theorem 3.1 because of the equality:

$$\Phi_L(gr^L(h(I))) = gr^L(I)$$

Let us prove this equality. First let $P \in I$ be of order $ord_L(P) = d$. We have:

$$h(P) = \sum_{\alpha,\beta} p_{\alpha,\beta} t^{d-|\beta|} x^\alpha D^\beta$$

$$\sigma^L(h(P)) = \sum_{L(\alpha,\beta)=d} p_{\alpha,\beta} t^{d-|\beta|} x^\alpha D^\beta.$$

The operation $t = 1$ then clearly gives:

$$\Phi_L(\sigma^L(h(P))) = \sum_{L(\alpha,\beta)=d} p_{\alpha,\beta} x^\alpha D^\beta = \sigma^L(P)$$

This proves that $\sigma^L(P) \in \Phi_L(gr^L(h(I)))$.

The other inclusion is less obvious: Let $R = \sum Q_i h(P_i)$ with $P_i \in I$ be an homogeneous element of degree d, for the graduation by (t, D) . If P_i has degree d_i, this means that Q_i is homogeneous with $d - d_i$. Then $Q_i = h(Q_i(1))t^{d-d_i-e_i}$ where e_i is the degree of the operator $Q_i(1)$. Thus we have $R = \sum t^{d-d_i-e_i} h(Q_i(1)P_i) = t^\lambda h(\sum Q_i(1)P_i)$ for some λ.

This proves that $R = t^\lambda h(S)$ for an element S of I, and that $\sigma^L(R) = t^\lambda \sigma^L(h(S))$ in $\mathrm{gr}^L(\mathcal{D}_n[t])$. By the operation $t = 1$ this gives $\Phi_L(\sigma^L(R)) = \Phi_L(\sigma^L(h(S))) = \sigma^L(S)$ by the first part of the proof and it is an element of $\mathrm{gr}^L(I)$ as required. □

6 The standard fan

Let I be a non-zero left ideal of $\mathcal{D}$ and let $h(I)$ be the homogenized ideal of I in $\mathcal{D}[t]$. The purpose of this section is to study the stability of $\mathrm{gr}^L(h(I))$ and $\mathrm{gr}^L(I)$ when L varies in $\mathcal{U}$.

For all $E \subseteq \mathbf{N}^{1+2n}$ such that $E + \mathbf{N}^{1+2n} = E$, we set:

$$\mathcal{U}_E = \{L \in \mathcal{U}; \mathrm{Exp}_L(h(I)) = E\}.$$

With these notations we have the following:

Theorem 6.1 *There exists a partition $\mathcal{E}$ of $\mathcal{U}$ into convex rational polyhedral cones, such that for all element $\sigma \in \mathcal{E}$, $\mathrm{gr}^L(I)$ and $\mathrm{Exp}_L(I)$ do not depend on $L \in \sigma$. This partition is exactly the partition into the set on which $\mathrm{gr}^L(h(I))$ is fixed. Furthermore, every $\mathcal{U}_E$ is convex and a union of cones of $\mathcal{E}$.*

We start by fixing some notations. Let E be a subset of $\mathbf{N}^{2n+1}$ such that $E + \mathbf{N}^{2n+1} = E$ and let $L \in \mathcal{U}_E$. Then consider the reduced L-standard basis $Q_1, \ldots, Q_r$ of $h(I)$. By 5.3, $\{Q_1, \ldots, Q_r\}$ is also the reduced L'-standard basis, for all $L' \in \mathcal{U}_E$. Denote by $\sim$ the equivalence relation on $\mathcal{U}$ defined from $Q_1, \ldots, Q_r$ by:

$$L \sim L' \iff \sigma^L(Q_k) = \sigma^{L'}(Q_k) \quad \text{for} \quad \text{all} \quad k = 1, \ldots, r.$$

Lemma 6.2 *$\sim$ defines on $\mathcal{U}$ a finite partition into convex rational polyhedral cones and $\mathcal{U}_E$ is a union of a part of these cones.*

Proof. Let $L_1, L_2 \in \mathcal{U}$ such that $L_1 \sim L_2$ and let L in the segment $[L_1, L_2]$, also let $\theta \in [0, 1]$ such that $L = \theta \cdot L_1 + (1 - \theta) \cdot L_2$. Write for all $1 \le k \le r$, $Q_k = \widetilde{Q_k} + R_k$ with $\mathcal{N}(\widetilde{Q_k}) = \mathcal{N}(\sigma^{L_1}(Q_k)) = \mathcal{N}(\sigma^{L_2}(Q_k))$ and $\mathrm{ord}^{L_i}(R_k) < \mathrm{ord}^L(Q_k)$. Since for all $(\alpha, \beta) \in \mathbf{N}^{2n}, L(\alpha, \beta) = \theta \cdot L_1(\alpha, \beta) + (1 - \theta) \cdot L_2(\alpha, \beta)$, then $\sigma^L(Q_k) = \sigma^{L_1}(Q_k) = \sigma^{L_2}(Q_k)$ by an immediate verification. Therefore the equivalence classes are cones.

On the other hand, if $L_1 \sim L_2$ and $L_1 \in \mathcal{U}_E$, then $L_2 \in \mathcal{U}_E$. Indeed this equivalence implies $\exp_{L_2}(Q_k) = \exp_{L_1}(Q_k)$ for all k. Hence $\mathrm{Exp}_{L_2}(h(I)) \supset E$. The opposite inclusion follows from the division theorem 4.1 because this division depends only on the exponents. This proves that $\mathcal{U}_E$ is a union of classes for $\sim$, the number of classes being finite by 5.1. □

Proof *of Theorem 6.1.* We define $\mathcal{E}$ as follows: for each E we consider the restriction $\mathcal{E}_E$ to $\mathcal{U}_E$ of the above partition and then $\mathcal{E}$ is the finite union of the $\mathcal{E}_E$'s. On each cone of the partition, $\mathrm{gr}^L(h(I))$ and $\mathrm{Exp}_L(h(I))$ are fixed, and the same is true for $\mathrm{gr}^L(I)$ and $\mathrm{Exp}_L(I)$ thanks to the proof of the Theorems 2.1 and 2.3.

For the converse, we remark first that $\mathrm{Exp}_L(h(I)) = \mathrm{Exp}_L(\mathrm{gr}^L(h(I)))$ and that if $\{Q_1, \ldots, Q_r\}$ is the reduced L- standard basis for $h(I)$ then $\{\sigma^L(Q_1), \ldots, \sigma^L(Q_r)\}$ is the reduced L-standard basis of $\mathrm{gr}^L(h(I))$ (because the Q_i are homogeneous). Assume then that L' is another form such that $\mathrm{gr}^L(h(I)) = \mathrm{gr}^{L'}(h(I))$ without an a priori hypothesis on the set of exponents. This equality implies in particular that the rings $\mathrm{gr}^L(\mathcal{D}[t])$ and $\mathrm{gr}^{L'}(\mathcal{D}[t])$ are canonically isomorphic (of the same type) and admit a (L, L')-bigraduation. The ideal $\mathrm{gr}^L(h(I)) = \mathrm{gr}^{L'}(h(I))$ is (L, L')-bihomogeneous which implies that $\sigma^L(Q_k)$ is also bihomogeneous: in fact, consider the L'-component H_k of $\sigma^L(Q_k)$ containing the L-exponent. Then we have $H_k \in \mathrm{gr}^{L'}(h(I)) = \mathrm{gr}^L(h(I))$. By the unicity of the reduced L-standard basis of $\mathrm{gr}^L(h(I))$ this gives $H_k = \sigma^L(Q_k)$. We end the proof of the lemma as follows: There is

$Q'_k \in h(I)$ such that $H_k = \sigma^{L'}(Q'_k)$ by which we get $\exp_L(Q_k) = \exp_{L'}(Q'_k)$, hence $\mathrm{Exp}_{L'}(h(I)) \supset \mathrm{Exp}_L(h(I))$. By exchanging the roles of L and L' we get the equality $\mathrm{Exp}_{L'}(h(I)) \supset \mathrm{Exp}_L(h(I))$ and finally $\sigma^L(Q_k) = \sigma^{L'}(Q_k)$ for all k, by the unicity of the reduced standard basis.

This ends the proof of the theorem except for the convexity of $\mathcal{U}_E$ proved below:

Lemma 6.3 *$\mathcal{U}_E$ is a convex set: If $L_1, L_2 \in \mathcal{U}_E$, then $[L_1, L_2] \subseteq \mathcal{U}_E$.*

Proof. Let $L \in]L_1, L_2[$ and let $\theta \in]0,1[$ such that $L = \theta \cdot L_1 + (1-\theta) \cdot L_2$. For all $1 \leq k \leq r$, if $\exp_{L_1}(Q_k) = \exp_{L_2}(Q_k) = \exp_{L_1}(q_k(t)x^\alpha D^\beta)$, and if $Q_k = q_k(t)x^\alpha D^\beta + R_k$, then either $R_k = 0$, or $\mathrm{ord}^L(q_k(t)x^\alpha D^\beta) \geq \mathrm{ord}^L(R_k)$. Furthermore, if $\mathrm{ord}^L(q_k(t)x^\alpha D^\beta) = \mathrm{ord}^L(R_k)$, then $\mathrm{ord}^{L_i}(q_k(t)x^\alpha D^\beta) = \mathrm{ord}^{L_i}(R_k)$ for at least one $1 \leq i \leq 2$. In particular $\exp_L(Q_k) = \exp_{L_1}(q_k(t)x^\alpha D^\beta)$, which implies that $E \subseteq \mathrm{Exp}_L(h(I))$, and consequently, by division, that $E = \mathrm{Exp}_L(h(I))$, i.e. $L \in \mathcal{U}_E$. □ □

Definition 6.4 *$\mathcal{E}$ is called the standard fan of $h(I)$, which would be in the terminology of [17], the extended Gröbner fan of I.*

In what follows, we give some specifications about the partition:

The subspace $\mathcal{U}$ of $\mathbf{R}^n$ being defined by the equations $e_i + f_i \geq 0$ and $e_i \leq 0$ is naturally stratified as a subvariety with corners: there is a stratum $\mathcal{U}_{\epsilon,\eta}$ indexed by each $2n$-uple $(\epsilon_1, \eta_1, \dots, \epsilon_n, \eta_n)$ of elements of $\{0,1\}$, with $\epsilon_i = 0$ (resp. $\eta_i = 0$) if and only if $e_i + f_i = 0$ (resp. $e_i = 0$). For $L \in \mathcal{U}_{\epsilon,\eta}$ the ring $\mathrm{gr}^L\mathcal{D}$, which is of a fixed type, admits a multigraduation, with one graduation for each element of the $2n$-uple $(\epsilon_1, \eta_1, \dots, \epsilon_n, \eta_n)$ equal to 1. Each of these graduations are either a graduation by the form $\beta_i - \alpha_i$, that is by the V-degree in (x_i, D_i) or (x_i, ξ_i), or by the form β_i, that is by the degree in ξ_i. Of course the number of strata is 2^{2n}.

Definition 6.5 *We say that $\mathrm{gr}^L(h(I))$ is a multihomogeneous ideal of type $(\epsilon_1, \eta_1, \dots, \epsilon_n, \eta_n)$ if for any $H \in \mathrm{gr}^L(h(I))$, each multi-homogeneous component of H is also in $\mathrm{gr}^L(h(I))$.*

Lemma 6.6 *The ideal $\mathrm{gr}^L(h(I))$ is multi-homogeneous if and only if the reduced L-standard basis of $\mathrm{gr}^L(h(I))$ is multi-homogeneous.*

Proof. We use exactly the same argument as to obtain the equality $H_k = \sigma^L(Q_k)$ in the proof of the theorem 6.1 □

Proposition 6.7 *The set of forms $L \in \mathcal{U}_{\epsilon,\eta}$ such that $\mathrm{gr}^L(h(I))$ is multihomogeneous of the type ϵ, η is the union of the cones of $\mathcal{E}$ which are (relatively) open in $\mathcal{U}_{\epsilon,\eta}$.*

The following lemma is the essential step in the proof of this proposition.

Lemma 6.8 *Let $P \in h(I)$ be such that $\sigma^L(P)$ is multi-homogeneous: Then there is a neighborhood $\mathcal{W}$ in $\mathcal{U}_{\epsilon,\eta}$ such that for any $L' \in \mathcal{W}$, $\sigma^{L'}(P) = \sigma^L(P)$.*

Proof. There is a finite list of linear forms $L_1, \dots, L_p$ with $L_i(\alpha, \beta) = \beta_i - \alpha_i$ for $i = 1, \dots, q$ and $L_j(\alpha, \beta) = \beta_j$ for $i = q+1, \dots, p$, such that the elements of $\mathcal{U}_{\epsilon,\eta}$ are the linear combinations of $L_1, \dots, L_p$ with coefficients in $\mathbf{R}_+^\star$. We write $L_\mu = \sum_{i=1}^p \mu_i L_i$ for a general element of $\mathcal{U}_{\epsilon,\eta}$ ($\mu_i > 0$), and $L = L_\lambda$ with $\lambda_i > 0$ for all i.

The second result in proposition 5.1, means that the Newton diagrams $\mathcal{N}(\sigma^{L'}(P))$ can take only a finite number of values. Let $\mathcal{N}_0, \dots, \mathcal{N}_c$ be their list, with $\mathcal{N}_0 = \mathcal{N}(\sigma^L(P))$: Since $\sigma^L(P)$ is multi-homogeneous, $\mathcal{N}_0$ is minimal, for the inclusion order, in the set $\mathcal{N}_0, \dots, \mathcal{N}_c$, and we can write $L_i(\mathcal{N}_0) = A_i$ as the value of $L_i(\alpha, \beta)$ for any $(l, \alpha, \beta) \in \mathcal{N}_0$.

Choose then $(l^b, \alpha^b, \beta^b) \in \mathcal{N}_b \setminus \mathcal{N}_0$ for $b = 1, \dots, c$. We have:

$$L(\alpha^b, \beta^b) < L(\mathcal{N}_0) = \lambda_1 A_1 + \ldots + \lambda_p A_p$$

and, for μ sufficiently near to λ, we get by continuity $L_\mu(\alpha^b, \beta^b) < L_\mu(\mathcal{N}_0) = \mu_1 A_1 + \ldots + \mu_p A_p$. The Newton diagram of $\sigma^{L_\mu}(P)$ contains none of the exponents (l^b, α^b, β^b) for $b = 1, \dots, c$ therefore is equal to $\mathcal{N}_0$, and $\sigma^{L_\mu}(P) = \sigma^L(P)$ for $|\lambda_i - \mu_i|$ small enough as required. □

Proof *of the proposition 6.7.* Assume first that $\mathrm{gr}^L(h(I))$ is multi-homogeneous and that $\{Q_1, \ldots, Q_r\}$ is a reduced L-standard basis of $h(I)$. By applying the lemma to all the Q_k at the same time, we find that, when all the $|\lambda_i - \mu_i|$ are small enough, we have for any k the equality $\sigma^{L_\mu}(Q_k) = \sigma^L(Q_k)$. This proves that all these L_μ are in the same cone of $\mathcal{E}$ as L, by lemma 6.2. This ends the proof of the fact that this cone is relatively open in $\mathcal{U}_{\epsilon,\eta}$.

Conversely, assume that this cone is open. Then, for all nearby μ, $\sigma^{L_\mu}(Q_k) = \sigma^L(Q_k)$ by the definition of this cone so that with the notations of the lemma:

$$\forall (k, \alpha, \beta), (l, \gamma, \delta) \in \mathcal{N}(\sigma^L(P)) \; \forall \mu = \lambda + \rho, \quad \text{with } \rho \text{ small enough } :$$

$$L(\alpha, \beta) = L(\gamma, \delta)$$

$$\sum \rho_i (L_i(\alpha, \beta) - L_i(\gamma, \delta)) = 0.$$

Therefore for any i, $L_i(\alpha, \beta) = L_i(\gamma, \delta)$ which proves that $\sigma^L(P)$ is multi-homogeneous and ends the proof of the proposition. □

Remarks 6.9 *1) For $(\epsilon, \eta) = (1, 1)$, we obtain the open cones in $\mathcal{E}$ whose union is the set of L such that $\mathrm{gr}^L(h(I))$ is generated by monomials in the commutative ring $\mathbf{C}[x, \xi]$*

For $(\epsilon, \eta) = (1, 0)$ the ring $\mathrm{gr}^L(\mathcal{D}[t])$ is just the homogenization $A_n[t]$ of the Weyl algebra, and we obtain as the union of the relatively open cones on this stratum, the forms L such that $\mathrm{gr}^L(h(I))$ is torus fixed in the sense of [20], that is generated by operators of the form $x^a p(x_1\partial_1, \ldots, x_n\partial_n)\partial^b$, with $p \in \mathbf{C}[\theta_1, \ldots, \theta_n]$ and, for all i, $a_i b_i = 0$.

2) In the case of the Weyl algebra, we have a similar result but with only 2^n strata, and the proof is slightly simpler because all the $\mathcal{N}(Q_k)$ are finite (cf. [3])

3) With the same proof as in the proposition 6.7, we can see that if $\mathrm{gr}^{L_1}(h(I)) = \mathrm{gr}^{L_2}(h(I))$, then all the L in an open segment containing L_1 and L_2 have the same $\mathrm{gr}^L(h(I))$ i.e. are in the same cone of $\mathcal{E}$. This establishes the fact that each cone of $\mathcal{E}$ is (relatively) open in the affine space which it generates.

4) Similarly if we consider on the line $[L_1, L_2]$ a linear form L such that $\mathrm{gr}^L(h(I))$ is bi-homogeneous, then there is a neighborhood of L included in the cone of L (i.e. with the same $\mathrm{gr}^L(h(I))$). This is a way to recover the "twin lemmas" of [2], therefore to read the algebraic slopes of I on the standard fan.

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