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Locally quasi-homogeneous free divisors are Koszul free

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Abstract

Let X be a complex analytic manifold and $D \subset X$ a free divisor. If D is locally quasi-homogeneous, then the logarithmic de Rham complex associated to D is quasi-isomorphic to $\mathbf{R}j_*(\mathbb{C}_{X\setminus D})$, which is a perverse sheaf [4]. On the other hand, the logarithmic de Rham complex associated to a *Koszul* free divisor is perverse [2]. In this paper we prove that every locally quasi-homogeneous free divisor is Koszul free.

Résumé

Soit X une varieté analytique complexe et D un diviseur libre. Si D est localement casi-homogène, alors le complexe de de Rham logarithmique est casi-isomorphe à $\mathbf{R}j_*(\mathbb{C}_{X\setminus D})$, qui est un faisceau pervers [4]. D'un autre coté, le complexe de de Rham logarithmique associé à un diviseur *Koszul* libre est pervers [2]. Dans cet article nous démontrons que tout diviseur libre localement casi-homogène est Koszul libre.

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1 Introduction

Let X be a complex analytic manifold. For $D \subset X$ a divisor, let us write $j : U = X \setminus D \hookrightarrow X$ for the corresponding open inclusion and $\Omega^\bullet(*D)$ for the meromorphic de Rham complex with poles along D. In [6], Grothendieck proved that the canonical morphism $\Omega^\bullet(*D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism (in the derived category). This result is usually known as (a version of) *Grothendieck's Comparison Theorem*.

In [10], K. Saito introduced the subcomplex $\Omega_X^\bullet(\log D)$ of $\Omega^\bullet(*D)$, that he called *logarithmic de Rham complex* associated to D, generalising the well known case of normal crossing divisors (cf. [5]). In the same paper, K. Saito also introduced the important notion of *free divisor*.

In [4], it is proved that the logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ computes the cohomology of the complement U if D is a locally quasi-homogeneous free divisor (we say that D satisfies the *logarithmic comparison theorem*). In other words, the canonical morphism $\Omega_X^\bullet(\log D) \to \mathbf{R}j_*(\mathbb{C}_U)$ is an isomorphism, or using Grothendieck's result, the inclusion $\Omega_X^\bullet(\log D) \hookrightarrow \Omega^\bullet(*D)$ is a quasi-isomorphism. In fact, in [3] it is proved that, in the case of $\dim X = 2$, D is locally quasi-homogeneous if and only if it satisfies the logarithmic comparison theorem.

As the derived direct image $\mathbf{R}j_*(\mathbb{C}_U)$ is a perverse sheaf (it is the de Rham complex of the holonomic module of meromorphic functions with poles along D [8], II, th. 2.2.4), we deduce that the logarithmic comparison theorem for a free divisor D implies that the logarithmic de Rham complex associated to D is a perverse sheaf.

On the other hand, the first author proved in [2] the following results:
Let $D \subset X$ be a Koszul free divisor (see definition 2.3) and $\mathcal{I}$ the left ideal of the ring $\mathcal{D}_X$ of differential operators on X generated by the logarithmic vector fields with respect to D. Then:
1) The left $\mathcal{D}_X$-module $\mathcal{D}_X/\mathcal{I}$ is holonomic.
2) There is a canonical isomorphism in the derived category

$$\Omega_X^\bullet(\log D) \simeq \mathbf{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{I}, \mathcal{O}_X).$$

As a consequence of these results, the logarithmic de Rham complex associated to a Koszul free divisor is a perverse sheaf.

In this paper we prove the following result, suggested by the previous ones: every locally quasi-homogeneous free divisor is Koszul free (see theorem 3.2).

At the end we study some examples in dimension 2 and 3.

2 Preliminary results

Let X be a n-dimensional complex analytic manifold. We denote by $\pi : T^*X \to X$ the cotangent bundle, $\mathcal{O}_X$ the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of linear differential operators on X (with holomorphic coefficients), $\mathcal{G}r_{F^\bullet}(\mathcal{D}_X)$ the graduated ring associated to the filtration by the order and $\sigma(P)$ the principal symbol of a differential operator P. We will note $\mathcal{O} = \mathcal{O}_{X,x}$, $\mathcal{D} = \mathcal{D}_{X,x}$ and $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}r_{F^\bullet}(\mathcal{D}_X)_x$ the respective stalks at x, with x a point in X. Let $D \subset X$ a hypersurface, we denote by $\mathcal{D}\mathrm{er}(\log D)$ the $\mathcal{O}_X$-module of the logarithmic vector fields with respect to D [10].

Definition 2.1.– A divisor D is Euler-homogeneous at x if there is a local equation h for D around x, and a germ of logarithmic vector field δ such that $\delta(h) = h$.

The set of points where a divisor is Euler-homogeneous is open.

Definition 2.2.– (cf. [4]) A divisor D in a n-dimensional complex manifold X is locally quasi-homogeneous if at each point $q \in D$, there are local coordinates $(U; x_1, \ldots, x_n)$ centered at q (i.e. with $x_i(q) = 0$ for $i = 1, \ldots, n$) with respect to which $D \cap U$ has a weighted homogeneous defining equation (with strictly positive weights).

Obviously a locally quasi-homogeneous divisor is Euler-homogeneous at every point.

Definition 2.3.– ([2], def. 4.1.1) Let $D \subset X$ be a divisor. We say that D is a Koszul free divisor at x if it is free at x and there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $\mathcal{D}\mathrm{er}(\log D)_x$ such that the sequence of symbols $\{\sigma(\delta_1), \ldots, \sigma(\delta_n)\}$ is regular in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{G}r_{F^\bullet}(\mathcal{D}_X)_x$. If D is a Koszul free divisor at each point of D, we simply say that it is a Koszul free divisor.

Remark 2.4.– The ideal $I_{D,x} = \mathrm{Gr}_{F^\bullet}(\mathcal{D})\mathcal{D}\mathrm{er}(\log D)_x$ is generated by the elements of any basis of $\mathcal{D}\mathrm{er}(\log D)_x$. As D is Koszul free at x if and only if $\mathrm{depth}(I_{D,x}, \mathrm{Gr}_{F^\bullet}(\mathcal{D})) = n$ (cf. [7], cor. 16.8), it is clear that the definition of Koszul free divisor does not depend on the election of a particular basis. By the coherence of $\mathcal{G}r_{F^\bullet}(\mathcal{D}_X)$, if a divisor is Koszul free at a point, then it is Koszul free near that point.

We have not found a reference for the following well known proposition (see [7], th. 17.4 for the local case).

Proposition 2.5.– Let $\mathbb{C}\{x\}$ be the ring of convergent power series in the variables $x = x_1, \ldots, x_n$ and let G be the graded ring of polynomials in the variables $\xi_1, \ldots, \xi_t$ with coefficients in $\mathbb{C}\{x\}$. A sequence $\sigma_1, \ldots, \sigma_s$ of homogeneous polynomials in G is regular if and only if the set of zeros $V(I)$ of the ideal I generated by $\sigma_1, \ldots, \sigma_s$ has dimension $n+t-s$ in $U \times \mathbb{C}^t$, for some open neighborhood U of 0 (then each irreducible component has dimension $n+t-s$).

Proof: Let $\mathbb{C}\{x, \xi\}$ be the ring of convergent power series in the variables $x_1, \ldots, x_n, \xi_1, \ldots, \xi_t$. As the σ_i are homogeneous and the ring $\mathbb{C}\{x, \xi\}$ is a flat extension of G, the σ_i are a regular sequence in G if and only if they are a regular sequence in $\mathbb{C}\{x, \xi\}$. But the last condition is equivalent to the equality (*loc. cit.*):

$$\dim_{(0,0)}(V(I)) = \dim\left(\mathbb{C}\{x, \xi\}/I\right) = n + t - s.$$

Finally, using the fact that all the σ_i are homogeneous in the variables ξ, the local dimension of $V(I)$ at $(0,0)$ coincides with its dimension in $U \times \mathbb{C}^t$ for some neighborhood U of 0. C.Q.D.

Corollary 2.6.– Let $D \subset X$ be a free divisor. Let J be the ideal in $\mathcal{O}_{T^*X}$ generated by $\pi^{-1}\mathcal{D}\mathrm{er}(\log D)$. Then, D is Koszul free if and only if the set $V(J)$ of zeros of J has dimension n (in this case, each irreducible component of $V(J)$ has dimension n).

Proposition 2.7.– Let X be a complex manifold of dimension n and let $D \subset X$ be a divisor. Then:

1. Let $X' = X \times \mathbb{C}$ and $D' = D \times \mathbb{C}$. The divisor $D \subset X$ is Koszul free if and only if $D' \subset X'$ is Koszul free.
2. Let Y be another complex manifold of dimension r and let $E \subset Y$ be a divisor. Then: a) The divisor $(D \times Y) \cup (X \times E)$ is free if $D \subset X$ and $E \subset Y$ are free.
   b) The divisor $(D \times Y) \cup (X \times E)$ is Koszul free if $D \subset X$ and $E \subset Y$ are Koszul free.

Proof:

1. It is a consequence of [4], lemma 2.2, (iv) and the fact that $\sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X,p}[\xi_1, \ldots, \xi_n]$ if and only if $\xi_{n+1}, \sigma_1, \ldots, \sigma_n$ is a regular sequence in $\mathcal{O}_{X',(p,t)}[\xi_1, \ldots, \xi_n, \xi_{n+1}]$.

2. a) It is an immediate consequence of Saito's Criterion (cf. [4], lemma 2.2, (v)).
   b) It is a consequence of a) and Corollary 2.6.

C.Q.D.

Example 2.8.– Examples of Koszul free divisors are:
1) Nonsingular divisors.
2) Normal crossing divisors.
3) Plane curves: If $\dim_{\mathbb{C}} X = 2$, we know that every divisor $D \subset X$ is free [10], cor. 1.7. Let $\{\delta_1, \delta_2\}$ be a basis of $\mathcal{D}\mathrm{er}(\log D)_x$. Their symbols $\{\sigma_1, \sigma_2\}$ are obviously linearly independent over $\mathcal{O}$, and by Saito's Criterion [10], 1.8, they are relatively primes in $\mathrm{Gr}_{F^\bullet}(\mathcal{D}) = \mathcal{O}[\xi_1, \xi_2]$. So they form a regular sequence in $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$, and D is Koszul free (see [2], cor. 4.2.2).
4) Proposition 2.7 gives a way to obtain Koszul free divisors in any dimension.
5) There are irreducible Koszul free divisors Y in dimensions greater than 2, which are not normal crossing and do not have non trivial factors [9]: $X = \mathbb{C}^3$ and $Y \equiv \{f = 0\}$, with

$$f = 2^8 z^3 - 2^7 x^2 z^2 + 2^4 x^4 z + 2^4 3^2 x y^2 z - 2^2 x^3 y^2 - 3^3 y^4.$$

A basis of $\mathcal{D}\mathrm{er}(\log f)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{array}{ccccccccc} \delta_1 & = & 6y & \partial_x & + & (8z - 2x^2) & \partial_y & - & xy & \partial_z, \\ \delta_2 & = & (4x^2 - 48z) & \partial_x & + & 12xy & \partial_y & + & (9y^2 - 16xz) & \partial_z, \\ \delta_3 & = & 2x & \partial_x & + & 3y & \partial_y & + & 4z & \partial_z, \end{array}$$

and the sequence $\{\sigma(\delta_1), \sigma(\delta_2), \sigma(\delta_3)\}$ is $\mathrm{Gr}_{F^\bullet}(\mathcal{D})$-regular.

3 Main results

Proposition 3.1.– Let D be a free divisor in some analytic manifold X and let $\Sigma \subset D$ a discrete set of points. If D is Koszul free at every point $x \in D \setminus \Sigma$, then D is Koszul free (at every point of D).

Proof: Let $p \in \Sigma$ and let $\{\delta_1, \ldots, \delta_n\}$ be a basis of the logarithmic derivations of D at p. By corollary 2.6, we have to prove that the symbols

$\sigma_i = \sigma(\delta_i)$ define an analytic set $V = V(\sigma_1, \ldots, \sigma_n) \subset \pi^{-1}(U)$ of dimension $n = \dim X$, for some open neighborhood $U \subset X$ of p. Let U be an open neighborhood of p such that $U \cap \Sigma = \{p\}$. By hypothesis, we know that D is Koszul free in $U \setminus \{p\}$, and so (*loc. cit.*) the dimension of $V \cap \pi^{-1}(U \setminus \{p\}) = V \setminus T_p^* X$ is n. Now, let W be an irreducible component of V. It has, at least, dimension n. If W is contained in T_p^*X, then it must be equal to T_p^*X, and $\dim W = n$. If not, $\dim W = \dim(W \setminus T_p^*X) \leq \dim(V \setminus T_p^*X) = n$. So, we conclude that V has dimension n. C.Q.D.

Theorem 3.2.– Every locally quasi-homogeneous free divisor is Koszul free.

Proof: We proceed by induction on the dimension t of the ambient manifold X. For $t = 1$, the theorem is trivial and for $t = 2$, the theorem is directly proved in examples 2.8, 3). Now, we suppose that the result is true for $t < n$, and let D be a locally quasi-homogeneous free divisor of a complex analytic manifold X of dimension n. Let $p \in D$ and let $\{\delta_1, \ldots, \delta_n\}$ be a basis of the logarithmic derivations of D at p.

Thanks to [4], prop. 2.4 and lemma 2.2, (iv), there is an open neighborhood U of p such that for each $q \in U \cap D$, with $q \neq p$, the germ of pair (X, D, q) is isomorphic to a product $(\mathbb{C}^{n-1} \times \mathbb{C}, D' \times \mathbb{C}, (0,0))$, where D' is a locally quasi-homogeneous free divisor. Induction hypothesis implies that D' is a Koszul free divisor at 0. Then, by proposition 2.7.1., D is a Koszul free divisor at q too. We have then proved that D is a Koszul free divisor in $U \setminus \{p\}$. We conclude by using proposition 3.1. C.Q.D.

Corollary 3.3.– Every free divisor that is locally quasi-homogeneous at the complement of a discrete set, is Koszul free.

In particular, the last corollary gives rise a new proof of the fact that every divisor in dimension 2 is Koszul free (cf. 2.8, 3)).

4 Examples

We know several (related) kind of free divisors:

[LQH] Locally quasi-homogeneous (definition 2.2).

[EH] Euler homogeneous (definition 2.1).

[LCT] Free divisors satisfying the logarithmic comparison theorem.

[KF] Koszul free (definition 2.3).

[P] Free divisors such that the complex $\Omega_X^\bullet(\log D)$ is a perverse sheaf.

We have then the following implications:

[LQH] $\Rightarrow$ [EH] (obvious), [LQH] $\Rightarrow$ [LCT] by [4], th. 1.1,
[LCT] $\Rightarrow$ [P], by [8], II, th. 2.2.4) [KF] $\Rightarrow$ [P] by [2], th. 4.2.1,
[LQH] $\Rightarrow$ [KF] by theorem 3.2.

Example 4.1.– (Free divisors in dimension 2) We recall theorem 3.9 from [3]: Let X be a complex analytic manifold of dimension 2 and $D \subset X$ a divisor. The following conditions are equivalent:
1. D is Euler homogeneous.
2. D is locally quasi-homogeneous.
3. The logarithmic comparison theorem holds for D.
Consequently, in dimension 2 we have:

$$[\mathrm{LQH}] \Leftrightarrow [\mathrm{EH}] \Leftrightarrow [\mathrm{LCT}]$$

and [KF] and [P] always hold (cf. 2.8, 3)). In particular,

$$[\mathrm{KF}] \not\Rightarrow [\mathrm{LQH}], [\mathrm{EH}], [\mathrm{LCT}].$$

Examples of plane curves not satisfying logarithmic comparison theorem are, for instance, the curves of the family (cf. [3]):

$$x^q + y^q + xy^{p-1} = 0, \quad p \geq q+1 \geq 5.$$

Example 4.2.– (An example in dimension 3) Let consider $X = \mathbb{C}^3$ and $D = \{f = 0\}$, with $f = xy(x+y)(y+zx)$ [2]. A basis of $\mathcal{D}\mathrm{er}(\log D)$ is $\{\delta_1, \delta_2, \delta_3\}$, with

$$\begin{array}{rcl} \delta_1 &=& x\partial_x + y\partial_y, \\ \delta_2 &=& x^2\partial_x - y^2\partial_y - z(x+y)\partial_z, \\ \delta_3 &=& xz + y\partial_z, \end{array}$$

the determinant of the coefficients matrix being $-f$ and

$$\delta_1(f) = 4f, \quad \delta_2(f) = (2x-3y)f, \quad \delta_3(f) = xf.$$

In particular, D is Euler homogeneous and satisfies the logarithmic comparison theorem [3]. Let $I \subset \mathcal{O}_{T^*X}$ be the ideal generated by the symbols $\{\sigma_1, \sigma_2, \sigma_3\}$ of the basis of $\mathcal{D}\mathrm{er}(\log D)$. By corollary 2.6, D is not Koszul free, because the dimension of $V(I)$ at $((0,0,\lambda),0) \in T^*X$ is greater than 3. So, D is not locally quasi homogeneous neither.
So:

$$[\mathrm{LCT}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}], \qquad [\mathrm{EH}] \not\Rightarrow [\mathrm{KF}], [\mathrm{LQH}].$$

Finally, for the only relation that we have not solved, we quote the following conjecture from [3]:

Conjecture 4.3.– If the logarithmic comparison theorem holds for D, then D is Euler homogeneous.

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