Structure-preserving approximation of diffuse interface tumor growth models

D. Acosta-Soba*, F. Guillén-González† and J. R. Rodríguez-Galván*

This work is devoted to the numerical approximation of the a degenerate diffuse interface model for tumor growth. This model couples the Cahn-Hilliard equation for the tumor volume fraction u, a reaction-diffusion equation for the nutrient volume fraction n and the Darcy equation for the extracellular fluid velocity $\mathbf{v}$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain and $T > 0$. The model, introduced in [1], reads as follows: find $(\mathbf{v}, p, u, \mu_u, n)$ with $u, n \in [0,1]$ such that

$$
\begin{aligned}
&\text{(1a)} & \mathbf{v} &= -\nu K(\nabla p + u\nabla\mu_u + n\nabla\mu_n),\\
&\text{(1b)} & \nabla\cdot\mathbf{v} &= 0,\\
&\text{(1c)} & \partial_t u + \nabla\cdot(u\mathbf{v}) &= C_u\nabla\cdot(M(u)\nabla\mu_u)\\
& & &\quad + \delta P_0 P(u,n)(\mu_n - \mu_u)_\oplus,\\
&\text{(1d)} & \mu_u &= F'(u) - \varepsilon^2\Delta u - \chi_0 n,\\
&\text{(1e)} & \partial_t n + \nabla\cdot(n\mathbf{v}) &= C_n\nabla\cdot(M(n)\nabla\mu_n)\\
& & &\quad - \delta P_0 P(u,n)(\mu_n - \mu_u)_\oplus,
\end{aligned}
$$

in $\Omega \times (0,T)$, satisfying $u(0) = u_0$, $n(0) = n_0$ in Ω, with $u_0, n_0 \in L^2(\Omega)$ and $u_0, n_0 \in [0,1]$, and the following boundary conditions on $\partial\Omega \times (0,T)$,

$$
\text{(1f)} \quad \mathbf{v}\cdot\mathbf{n} = \nabla u\cdot\mathbf{n} = (M_n\nabla\mu_n)\cdot\mathbf{n} = (M_u\nabla\mu_u)\cdot\mathbf{n} = 0,
$$

where the parameters above are nonnegative with $\delta, C_u, C_n, K > 0$, $\varepsilon, \chi_0, P_0 \geq 0$ and $\nu \in \{0,1\}$.

Moreover, $F : \mathbb{R} \to \mathbb{R}$ is the Ginzburg-Landau double-well potential $F(u) = \frac{1}{4}u^2(1-u)^2$, $M(\cdot)$ is a degenerate mobility function, μ_u and μ_n are the chemical potentials for the tumor and nutrient variables, respectively, with $\mu_n := \frac{1}{\delta}n - \chi_0 u$ and $P(\cdot,\cdot)$ is a degenerate proliferation function. The operator $(\cdot)_\oplus$ denotes the positive part such that $(\phi)_\oplus = \max\{\phi, 0\}$.

In this talk, we introduce a structure-preserving discretization of the model (1) based on a upwind discontinuous Galerkin approximation in space and a semi-implicit scheme in time. The resulting discrete variables preserve the following properties of any solution $(\mathbf{v}, p, u, \mu_u, n)$ of the continuous model (1):

PROPOSITION 1 *The total mass of tumor cells and nutrients is conserved in the sense of* $\partial_t \int_\Omega (u(x,t) + n(x,t))dx = 0$.

PROPOSITION 2 *The following energy law is satisfied*

$$
\partial_t E(u,n) + C_u\int_\Omega M(u)|\nabla\mu_u|^2 + C_n\int_\Omega M(n)|\nabla\mu_n|^2 + \delta P_0\int_\Omega P(u,n)(\mu_u-\mu_n)^2_\oplus + \frac{1}{K}\int_\Omega |\mathbf{v}|^2 = 0,
$$

where the energy functional is defined by

$$
E(u,n) := \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + F(u) - \chi_0 u\, n + \frac{1}{2\delta}n^2\right).
$$

Moreover, we will also present some numerical results that illustrate the performance of the proposed scheme.

The details of the numerical approximation and the proofs of the results in the absence of the extracellular fluid interaction ($\nu = 0$) have been published in [2]. Currently, we are working on extending the ideas for the case $\nu = 1$ using the stabilization techniques developed in [3], direction in which we have made some promising progress that will be reported.

References

[1] D. Acosta-Soba, *Analysis and numerical simulation of tumor growth models*. Ph.D. thesis, 2024. `http://hdl.handle.net/10498/32685`

[2] D. Acosta-Soba, F. Guillén-González, J. R. Rodríguez-Galván, *A structure-preserving upwind DG scheme for a degenerate phase-field tumor model*. Computers & Mathematics With Applications, **152** pp. 317-333, 2023. `https://doi.org/10.1016/j.camwa.2023.10.028`

[3] D. Acosta-Soba, F. Guillén-González, J. R. Rodríguez-Galván, J. Wang, *Property-preserving numerical approximation of a Cahn–Hilliard–Navier–Stokes model with variable density and degenerate mobility*. Applied Numerical Mathematics, **209** pp. 68-83, 2025. `https://doi.org/10.1016/j.apnum.2024.11.005`

*Departamento de Matemáticas, Universidad de Cádiz, Facultad de Ciencias, Campus Universitario Río San Pedro s/n., 11510 Puerto Real, Cádiz (SPAIN). Email: daniel.acosta@uca.es, rafael.rodriguez@uca.es
†Departamento EDAN & IMUS, Universidad de Sevilla, Campus Reina Mercedes, 41012 Sevilla, (SPAIN). Email: guillen@us.es