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Boundary estimation for the Stokes system

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Neglecting the inertial term in the Navier–Stokes system leads to the Stokes system

$$\begin{cases} \partial_t U - \Delta U + \nabla q = F, \\ \operatorname{div} U = 0. \end{cases}$$

We are interested in observing this system from an interior region of a domain. We consider general boundary conditions that include, for instance, the commonly used Dirichlet, Navier, and Neumann conditions.

Observation is achieved through a *local Carleman inequality* near a boundary point. This inequality is derived from the full system, including the pressure term.

Carleman inequalities are weighted estimates where the weight is exponential. For the Laplace operator, it takes the form

$$(1) \qquad \tau^{3/2}\|e^{\tau\varphi}v\|_{L^2} + \tau^{1/2}\|e^{\tau\varphi}\nabla v\|_{L^2} \leq C\|e^{\tau\varphi}\Delta v\|_{L^2},$$

for a function v compactly supported. Here, the parameter $\tau > 0$ can be chosen as large as needed. The choice of the function φ is delicate and depends on the intended application of the estimate. For the Laplace operator, this is a *sub-elliptic estimate* with a loss of half a derivative, which is reflected by the term $\tau^{3/2}$ instead of τ^2 in front of the L^2-norm of v on the left-hand side of (1); see for instance [2]. Boundary conditions are also necessary to obtain an estimate as in (1) near the boundary.

For a parabolic operator, the weight function can be chosen as in the seminal work of Fursikov and Imanuvilov [1], that is, $\varphi(t,x) = t^{-1}(T-t)^{-1}\phi(x)$ and the estimation has the form

$$(2) \qquad \tau^3 \int_0^T \|\varphi^{3/2}e^{\tau\varphi}v\|_{L^2}^2 dt + \tau \int_0^T \|\varphi^{1/2}e^{\tau\varphi}\nabla v\|_{L^2}^2 dt \leq C \int_0^T \|e^{\tau\varphi}(\partial_t - \Delta)v\|_{L^2}^2 dt.$$

We begin by reviewing how boundary estimates can be obtained for first-order scalar operators. Then, by expressing the Stokes system in the form of a first-order system, we show how various scalar reductions of the Stokes system can lead to such first-order equations using eigenvectors and generalized eigenvectors. This analysis is carried out *microlocally* in different regions of phase space.

Boundary conditions are handled in the spirit of the treatment of Lopatinskiĭ-Šapiro conditions for the Laplace operator; see for instance [3].

We obtain an estimate where there is a loss of a full derivative for the velocity U, and a loss of half a derivative for the pression q. The loss of a full derivative makes the analysis sometimes intricate since this is a threshold that may prevent one to handle remainder terms that appear along some of the estimations. With a weight function as for (2), the final estimation we obtain is of the form

$$(3) \qquad \tau^2 \int_0^T \|\varphi e^{\tau\varphi}U\|_{L^2}^2 dt + \int_0^T \|e^{\tau\varphi}\nabla U\|_{L^2}^2 dt + \tau \int_0^T \|\varphi^{1/2}e^{\tau\varphi}q\|_{L^2}^2 dt \leq C \int_0^T \|e^{\tau\varphi}F\|_{L^2}^2 dt,$$

with additional estimations of the Dirichlet and Neumann traces of U and q.

This is joint work with Luc Robbiano (Université de Versailles Saint-Quentin).

References

[1] FURSIKOV, A., AND IMANUVILOV, O. Y. *Controllability of evolution equations*, vol. 34. Seoul National University, Korea, 1996. Lecture notes.

[2] J. Le Rousseau, G. Lebeau, and L. Robbiano. *Elliptic Carleman Estimates and Applications to Stabilization and Controllability, Volume I: Dirichlet Boundary Conditions on Euclidean Space*. PNLDE Subseries in Control. Birkhäuser, 2022.

[3] J. Le Rousseau, G. Lebeau, and L. Robbiano. *Elliptic Carleman Estimates and Applications to Stabilization and Controllability, Volume II: General Boundary Conditions on Riemnannian Manifolds*. PNLDE Subseries in Control. Birkhäuser, 2022.

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