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On the local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions

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Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded domain with boundary $\Gamma := \partial\Omega$ of class C^2. For $T > 0$, we define the sets $Q := \Omega \times (0,T)$, $\Sigma := \Gamma \times (0,T)$ and for any subset $\omega \subset \Omega$, we write $Q_\omega := \omega \times (0,T)$.

For $a, b > 0$ and $\alpha \in \mathbb{R}$, we define the linear operators

$$L(u) := \partial_t u - a(1+\alpha i)\Delta u$$
$$L_\Gamma(u, u_\Gamma) := \partial_t u_\Gamma + a(1+\alpha i)\partial_\nu u - b(1+\alpha i)\Delta_\Gamma u_\Gamma$$

Moreover, for $c, \gamma \in \mathbb{R}$ we put $f(w) := c(1+\gamma i)|w|^2 w_\Gamma$. Then, we consider the following system

$$\begin{cases} L(u) + f(u) = \mathbb{1}_\omega h & \text{in } Q, \\ L_\Gamma(u, u_\Gamma) + f(u_\Gamma) = 0 & \text{on } \Sigma, \\ u\big|_\Gamma = u_\Gamma & \text{on } \Sigma, \\ (u(\cdot,0), u_\Gamma(\cdot,0)) = (u_0, u_{\Gamma,0}) & \text{in } \Omega \times \Gamma. \end{cases} \tag{1}$$

The control h acts only on the first equation. This means that the second equation (i.e., the general dynamic boundary condition) is controlled by the side condition $u\big|_\Gamma = u_\Gamma$.

Let us define the spaces

$$\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma),$$
$$\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y\big|_\Gamma = y_\Gamma\}$$

It is not difficult to see that, under a smallness condition on both the initial data and the control, a unique solution of (1) exists.

Our main result is the following

THEOREM 1 *Suppose that $d = 2$ or $d = 3$ and $\omega \Subset \Omega$. Then, the system* (1) *is* **locally null controllable at every time** T *in* $\mathbb{H}^1$*, i.e., there exists $\delta > 0$ such that, for every $(u_0, u_{\Gamma,0}) \in \mathbb{H}^1$ verifying*

$$\|(u_0, u_{\Gamma,0})\|_{\mathbb{H}^1} \leq \delta,$$

there exists a control $h \in L^2(\omega \times (0,T))$ such that the solution (u, u_Γ) of (1) *satisfies*

$$u(\cdot, T) = 0 \text{ in } \Omega, \quad u_\Gamma(\cdot, T) = 0 \text{ on } \Gamma.$$

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References

[1] N. Carreño, Mercado, A., & R. Morales. *Local null controllability of a cubic Ginzburg-Landau equation with dynamic boundary conditions* (To appear in Journal of Evolution Equations) https://arxiv.org/abs/2301.03429

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