Nitsche's prescription of Dirichlet conditions for the conforming finite element approximation of Maxwell's problem

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Outline

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Problem setting

Consider Maxwell's problem with Dirichlet boundary conditions:

$$\begin{aligned} \nu\nabla\times\nabla\times\boldsymbol{u}+\nabla p &= \boldsymbol{f} && \text{in } \Omega, && (1)\\ -\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, && (2)\\ \boldsymbol{n}\times\boldsymbol{u} &= \boldsymbol{n}\times\bar{\boldsymbol{u}} && \text{on } \Gamma, && (3)\\ p &= \bar{p} := 0 && \text{on } \Gamma. && (4) \end{aligned}$$

Suppose that Ω is discretised using finite elements (FEs). We consider the following situation:

- Conforming approximation in $H(\mathrm{curl};\Omega)\times H_0^1(\Omega)$.
- Weak imposition of BCs using Nitsche's method.

Note that

- Nitsche's method is well understood for coercive problems, but Maxwell's problem is not coercive, only inf-sup stable.
- Discontinuous Galerkin approximations using Nitsche's method are well understood, but we consider conforming approximations.

Objectives

To provide stability and convergence results for Nitsche's method in appropriate norms using two different formulations:

- A Galerkin approach using conforming inf-sup stable approximations in $H(\mathrm{curl};\Omega) \times H_0^1(\Omega)$; for example Nédélec's elements for $\boldsymbol{u}$ and continuous Lagrangian elements for p.
- A stabilised FE formulation that allows to use arbitrary, but conforming, approximations for $\boldsymbol{u}$ and p; for example, continuous Lagrangian elements for both $\boldsymbol{u}$ and p.

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Variational problem

Consider the identity:

$$\begin{aligned}
&\langle \mathcal{L}([\boldsymbol{u},p]),[\boldsymbol{v},q]\rangle_\Omega = B([\boldsymbol{u},p],[\boldsymbol{v},q]) - \langle \mathcal{F}_n([\boldsymbol{u},p]),\mathcal{D}([\boldsymbol{v},q])\rangle_\Gamma,\\
&B([\boldsymbol{u},p],[\boldsymbol{v},q]) := \nu(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega + (\nabla p,\boldsymbol{v})_\Omega + (\nabla q,\boldsymbol{u})_\Omega,\\
&\mathcal{F}_n([\boldsymbol{u},p]) := [\nu P_t(\nabla\times\boldsymbol{u}),\boldsymbol{n}\cdot\boldsymbol{u}],\quad \mathcal{F}([\boldsymbol{u},p]) := [\nu\nabla\times\boldsymbol{u},\boldsymbol{u}],\\
&\mathcal{D}([\boldsymbol{v},q]) := [\boldsymbol{n}\times\boldsymbol{v},q].
\end{aligned}$$

The variational form of problem (1)-(4) is well posed in the space $V\times Q := H(\mathrm{curl};\Omega)\times H^1(\Omega)$. Taking for the moment $\bar{\boldsymbol{u}}=\boldsymbol{0}$, it reads: find $[\boldsymbol{u},p]\in V_0\times Q_0$ such that

$$B([\boldsymbol{u},p],[\boldsymbol{v},q]) = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega, \tag{5}$$

for all $[\boldsymbol{v},q]\in V_0\times Q_0$, i.e., $[\boldsymbol{v},q]\in V\times Q$ and $\mathcal{D}([\boldsymbol{v},q]) = [\boldsymbol{0},0]$.

Inf-sup conditions

The continuous problem is well posed because it can be shown that

$$\inf_{[\boldsymbol{u},p]\in V_0\times Q_0} \sup_{[\boldsymbol{v},q]\in V_0\times Q_0} \frac{B([\boldsymbol{u},p],[\boldsymbol{v},q])}{\|[\boldsymbol{u},p]\|_{V\times Q}\|[\boldsymbol{v},q]\|_{V\times Q}} \geq K_B > 0, \tag{6}$$

$$\|[\boldsymbol{u},p]\|^2_{V\times Q} := \|\boldsymbol{u}\|^2_V + \|p\|^2_Q,$$

$$\|\boldsymbol{u}\|^2_V := \nu\|\nabla\times\boldsymbol{u}\|^2_{L^2(\Omega)} + \frac{\nu}{L_0^2}\|\boldsymbol{u}\|^2_{L^2(\Omega)}, \quad \|p\|^2_Q := \frac{L_0^2}{\nu}\|\nabla p\|^2_{L^2(\Omega)}.$$

The continuous problem (5) is equivalent to the two variational equations:

$$a(\boldsymbol{u},\boldsymbol{v}) + b(p,\boldsymbol{v}) = \langle \boldsymbol{f},\boldsymbol{v}\rangle_\Omega \qquad \forall \boldsymbol{v}\in V_0, \tag{7}$$

$$b(q,\boldsymbol{u}) = 0 \qquad \forall q\in Q_0, \tag{8}$$

$$a(\boldsymbol{u},\boldsymbol{v}) := \nu(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v})_\Omega, \quad b(p,\boldsymbol{v}) := (\nabla p,\boldsymbol{v})_\Omega.$$

The inf-sup condition (6) is then a consequence of:

$$\inf_{p\in Q_0}\sup_{\boldsymbol{v}\in V_0}\frac{b(p,\boldsymbol{v})}{\|p\|_Q\|\boldsymbol{v}\|_V} \geq K_b > 0, \tag{9}$$

and the coercivity of $a(\boldsymbol{u},\boldsymbol{v})$ in $K_V := \{\boldsymbol{v}\in V_0 \mid b(q,\boldsymbol{v}) = 0\ \forall q\in Q_0\}$.

Galerkin FE approximation with essential BCs: spaces

We assume in this subsection that the discrete version of (9) holds:

$$\inf_{p_h \in Q_{h,0}} \sup_{\boldsymbol{v}_h \in V_{h,0}} \frac{b(p_h, \boldsymbol{v}_h)}{\|p_h\|_Q \|\boldsymbol{v}_h\|_V} \geq K_b > 0,$$

or, equivalently,

$$\forall p_h \in Q_{h,0} \; \exists \, \boldsymbol{v}_h \in V_{h,0} \text{ such that } b(p_h, \boldsymbol{v}_h) \geq K_b \|p_h\|_Q \|\boldsymbol{v}_h\|_V. \tag{10}$$

Examples of pairs of spaces satisfying this condition are those based on Nédélec's elements to construct $V_{h,0}$ and nodal Lagrangian continuous elements to construct $Q_{h,0}$. In general, condition (10) is guaranteed if the following diagram commutes:

$$\begin{tikzcd}
H_0^1(\Omega) \arrow[r, "\nabla"] \arrow[d, "P_{Q_h}"] & H_0(\mathrm{curl}; \Omega) \arrow[d, "P_{V_h}"] \\
Q_{h,0} \arrow[r, "\nabla"] & V_{h,0}
\end{tikzcd}$$

Galerkin FE approximation with essential BCs: main result

Theorem

Suppose that both $V_{h,0}$ and $Q_{h,0}$ are an inf-sup stable pair satisfying condition (10). Then, the Galerkin FE approximation is well posed, in the sense that it admits a unique solution $[\boldsymbol{u}_h, p_h] \in V_{h,0} \times Q_{h,0}$ that satisfies

$$\|[\boldsymbol{u}_h, p_h]\|_{V \times Q} \lesssim \|\boldsymbol{f}\|_{V'}.$$

Furthermore, $[\boldsymbol{u}_h, p_h]$ converges optimally as $h \to 0$ to the solution $[\boldsymbol{u}, p] \in V_0 \times Q_0$ of the continuous problem (7)-(8), in the following sense:

$$\|[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]\|_{V \times Q} \lesssim \inf_{[\boldsymbol{v}_h, q_h] \in V_{h,0} \times Q_{h,0}} \|[\boldsymbol{u} - \boldsymbol{v}_h, p - q_h]\|_{V \times Q}.$$

Stabilised FE approximation: the formulation

The original problem is equivalent to:

$$\begin{aligned} \nu \nabla \times \nabla \times \boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } \Omega, \\ \frac{L_0^2}{\nu} \Delta p - \nabla \cdot \boldsymbol{u} = 0 \qquad & \text{in } \Omega, \\ \boldsymbol{n} \times \boldsymbol{u} = \boldsymbol{n} \times \bar{\boldsymbol{u}} \quad & \text{on } \Gamma, \\ p = \bar{p} := 0 \quad & \text{on } \Gamma. \end{aligned}$$

We propose the following formulation: find $[\boldsymbol{u}_h, p_h] \in V_{h,0} \times Q_{h,0}$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(p_h, \boldsymbol{v}_h) + s_u(\boldsymbol{u}_h, \boldsymbol{v}_h) = \langle \boldsymbol{f}, \boldsymbol{v}_h \rangle_\Omega \quad \forall \boldsymbol{v}_h \in V_{h,0}, \tag{11}$$

$$b(q_h, \boldsymbol{u}_h) + s_p(p_h, q_h) = 0 \quad \forall q_h \in Q_{h,0}, \tag{12}$$

$$s_u(\boldsymbol{u}_h, \boldsymbol{v}_h) := c_u \frac{\nu h^2}{L_0^2} (\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_\Omega, \quad s_p(p_h, q_h) := -\frac{L_0^2}{\nu} (\nabla p_h, \nabla q_h)_\Omega.$$

Stabilised FE approximation: main result

Let

$$\|[\boldsymbol{v}_h, q_h]\|^2_{V\times Q,\mathrm{S}} := \|[\boldsymbol{v}_h, q_h]\|^2_{V\times Q} + \nu \frac{h^2}{L_0^2} \|\nabla \cdot \boldsymbol{u}_h\|^2_{L^2(\Omega)}.$$

Theorem

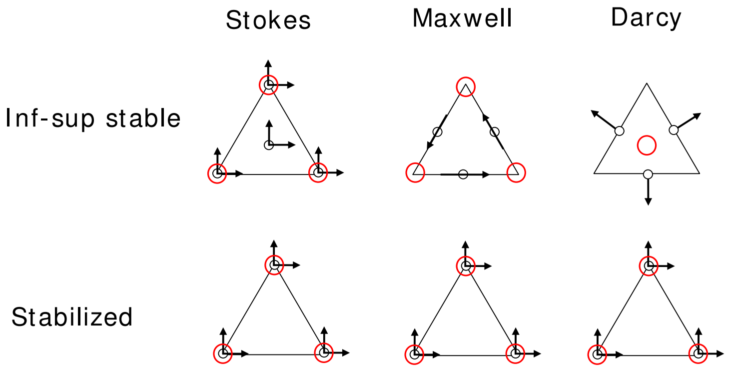
Suppose that both $V_{h,0}$ and $Q_{h,0}$ are constructed using continuous nodal based interpolations. Then, problem (11)-(12) is well posed: it admits a unique solution $[\boldsymbol{u}_h, p_h] \in V_{h,0} \times Q_{h,0}$ that satisfies

$$\|[\boldsymbol{u}_h, p_h]\|_{V\times Q} \lesssim \|\boldsymbol{f}\|_{V'}.$$

Furthermore, $[\boldsymbol{u}_h, p_h]$ converges optimally as $h \to 0$ to the solution $[\boldsymbol{u}, p] \in V_0 \times Q_0$ of the continuous problem (7)-(8):

$$\|[\boldsymbol{u} - \boldsymbol{u}_h, p - p_h]\|_{V\times Q,\mathrm{S}} \lesssim \inf_{[\boldsymbol{v}_h, q_h] \in V_{h,0}\times Q_{h,0}} \|[\boldsymbol{u} - \boldsymbol{v}_h, p - q_h]\|_{V\times Q,\mathrm{S}}.$$

Comparison of inf-sup stable vs stabilised interpolations for Stokes', Maxwell's and Darcy's problems



Outline

Galerkin approximation without BCs

If no BCs are imposed, one can show that:

Theorem

Suppose that the FE space $V_{h,0} \times Q_{h,0}$ satisfies the inf-sup condition (10). Then, for each $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ there exists $[\boldsymbol{v}_{h,0}, q_{h,0}] \in V_{h,0} \times Q_{h,0}$ such that

$$B([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_{h,0}, q_{h,0}]) \gtrsim \|[\boldsymbol{u}_h, p_h]\|^2_{V\times Q} - \gamma\frac{\nu}{h}\|\boldsymbol{n} \times \boldsymbol{u}_h\|^2_{L^2(\Gamma)} - \gamma\frac{L_0^2}{\nu h}\|p_h\|^2_{L^2(\Gamma)}, \tag{13}$$

for a constant $\gamma \geq 0$.

Estimate (13) explicitly displays which terms spoil stability of the problem without boundary conditions. The terms introduced by Nitsche's method need precisely to compensate them.

Nitsche's method for the Galerkin FE approximation: method

It reads: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_{\mathrm{N}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_{\mathrm{N}}([\boldsymbol{v}_h, q_h]) \qquad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h,$$

where

$$\begin{aligned} B_{\mathrm{N}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) := {} & \nu(\nabla \times \boldsymbol{v}_h, \nabla \times \boldsymbol{u}_h)_\Omega + (\boldsymbol{v}_h, \nabla p_h)_\Omega + (\boldsymbol{u}_h, \nabla q_h)_\Omega \\ & - \nu\langle \boldsymbol{n} \times \boldsymbol{v}_h, \nabla \times \boldsymbol{u}_h \rangle_\Gamma - \langle \boldsymbol{n} \cdot \boldsymbol{u}_h, q_h \rangle_\Gamma \\ & - \nu\langle \boldsymbol{n} \times \boldsymbol{u}_h, \nabla \times \boldsymbol{v}_h \rangle_\Gamma - \langle \boldsymbol{n} \cdot \boldsymbol{v}_h, p_h \rangle_\Gamma \\ & + N_u \frac{\nu}{h} \langle \boldsymbol{n} \times \boldsymbol{v}_h, \boldsymbol{n} \times \boldsymbol{u}_h \rangle_\Gamma - N_p \frac{L_0^2}{\nu h} (p_h, q_h)_\Gamma \end{aligned}$$

$$L_{\mathrm{N}}([\boldsymbol{v}_h, q_h]) := \langle \boldsymbol{v}_h, \boldsymbol{f} \rangle_\Omega - \nu\langle \boldsymbol{n} \times \bar{\boldsymbol{u}}, \nabla \times \boldsymbol{v}_h \rangle_\Gamma + N_u \frac{\nu}{h} \langle \boldsymbol{n} \times \boldsymbol{v}_h, \boldsymbol{n} \times \bar{\boldsymbol{u}} \rangle_\Gamma.$$

Nitsche's method for the Galerkin FE approximation: main result

Theorem

Assume that the FE space $V_{h,0} \times Q_{h,0}$ satisfies the inf-sup condition (10). Then, for N_u and N_p sufficiently large, B_{N} is inf-sup stable in the norm

$$|||[\boldsymbol{v}_h, q_h]|||^2_{V\times Q,\mathrm{N}} := \|[\boldsymbol{v}_h, q_h]\|^2_{V\times Q} + \frac{\nu}{h}\|\boldsymbol{n} \times \boldsymbol{v}_h\|^2_{L^2(\Gamma)} + \frac{L_0^2}{\nu h}\|q_h\|^2_{L^2(\Gamma)}.$$

Furthermore, $[\boldsymbol{u}_h, p_h]$ converges optimally in this norm as $h \to 0$ to the solution $[\boldsymbol{u}, p] \in V \times Q$ of the continuous problem (7)-(8).

Stabilised FE approximation without BCs

If no BCs are imposed, for the stabilised FE approximation it can be proved that

Theorem

For each $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ *there exists* $[\boldsymbol{v}_{h,0}, q_{h,0}] \in V_{h,0} \times Q_{h,0}$ *such that*

$$B_{\mathrm{S}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_{h,0}, q_{h,0}]) \gtrsim \|[\boldsymbol{u}_h, p_h]\|^2_{V\times Q,\mathrm{S}}$$
$$-\gamma \frac{\nu}{h} \|\boldsymbol{n} \times \boldsymbol{u}_h\|^2_{L^2(\Gamma)} - \gamma \frac{L_0^2}{\nu h} \|p_h\|^2_{L^2(\Gamma)},$$

for a constant $\gamma \geq 0$.

Nitsche's method for the stabilised FE approximation: method

It reads: find $[\boldsymbol{u}_h, p_h] \in V_h \times Q_h$ such that

$$B_{\mathrm{SN}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) = L_{\mathrm{N}}([\boldsymbol{v}_h, q_h]) \qquad \forall [\boldsymbol{v}_h, q_h] \in V_h \times Q_h,$$

where

$$\begin{aligned} B_{\mathrm{SN}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) &:= B_{\mathrm{S}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) \\ &\quad - \nu\langle \boldsymbol{n} \times \boldsymbol{v}_h, \nabla \times \boldsymbol{u}_h\rangle_\Gamma - \langle \boldsymbol{n} \cdot \boldsymbol{u}_h, q_h\rangle_\Gamma - \nu\langle \boldsymbol{n} \times \boldsymbol{u}_h, \nabla \times \boldsymbol{v}_h\rangle_\Gamma - \langle \boldsymbol{n} \cdot \boldsymbol{v}_h, p_h\rangle_\Gamma \\ &\quad + \frac{L_0^2}{\nu}\langle \boldsymbol{n} \cdot \nabla p_h, q_h\rangle_\Gamma + \frac{L_0^2}{\nu}\langle p_h, \boldsymbol{n} \cdot \nabla q_h\rangle_\Gamma + N_u \frac{\nu}{h}\langle \boldsymbol{n} \times \boldsymbol{v}_h, \boldsymbol{n} \times \boldsymbol{u}_h\rangle_\Gamma \\ &\quad - N_p \frac{L_0^2}{\nu h}(p_h, q_h)_\Gamma \\ &= B_{\mathrm{N}}([\boldsymbol{u}_h, p_h], [\boldsymbol{v}_h, q_h]) + c_u \frac{\nu h^2}{L_0^2}(\nabla \cdot \boldsymbol{u}_h, \nabla \cdot \boldsymbol{v}_h)_\Omega - \frac{L_0^2}{\nu}(\nabla p_h, \nabla q_h)_\Omega \\ &\quad + \frac{L_0^2}{\nu}\langle \boldsymbol{n} \cdot \nabla p_h, q_h\rangle_\Gamma + \frac{L_0^2}{\nu}\langle p_h, \boldsymbol{n} \cdot \nabla q_h\rangle_\Gamma. \end{aligned} \tag{14}$$

Nitsche's method for the stabilised FE approximation: main result

Theorem

Consider the stabilised bilinear form using Nitsche's method given by (14). Then, for N_u *and* N_p *sufficiently large,* B_{SN} *is inf-sup stable in the norm*

$$|||[\boldsymbol{v}_h, q_h]|||^2_{V\times Q,\mathrm{SN}} := |||[\boldsymbol{v}_h, q_h]|||^2_{V\times Q,\mathrm{S}} + \frac{\nu}{h}\|\boldsymbol{n}\times\boldsymbol{v}_h\|^2_{L^2(\Gamma)} + \frac{L_0^2}{\nu h}\|q_h\|^2_{L^2(\Gamma)},$$

Furthermore, $[\boldsymbol{u}_h, p_h]$ *converges optimally in this norm as* $h \to 0$ *to the solution* $[\boldsymbol{u}, p] \in V \times Q$ *of the continuous problem (7)-(8).*

Outline

The square domain I

Let $\Omega = (-1,1)^2$, with a smooth manufactured solution given by $\boldsymbol{u}(x,y) = (\varphi(x)\varphi'(y), -\varphi'(x)\varphi(y))$, where $\varphi(t) = t^2 \sin(\pi t/2)$.

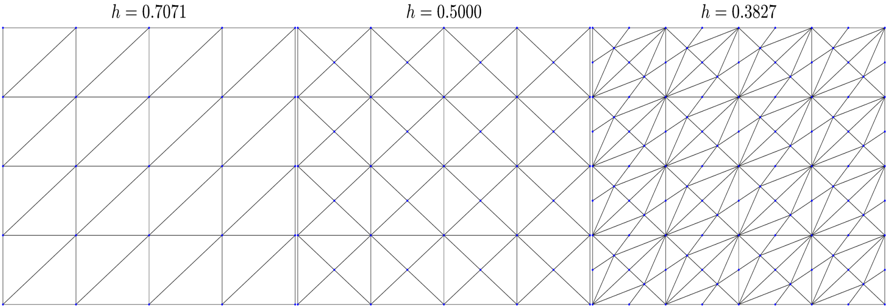


Figure: The standard uniform right-angled (L), criss-cross (M), and Powell-Sabin (R) meshing of the square domain.

The square domain II

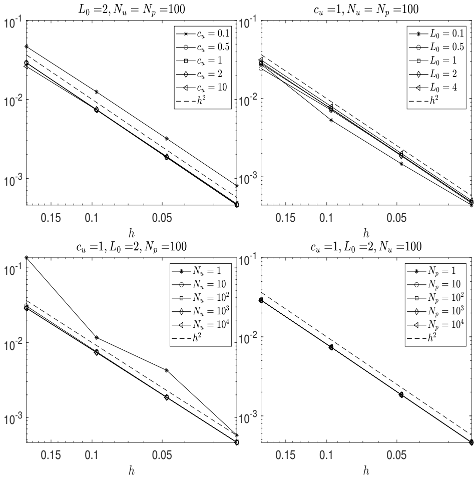


Figure: Parameter tests: c_u variation (top left), L_0 variation (top right), N_u variation (bottom left), and N_p variation (bottom right).

The square domain III

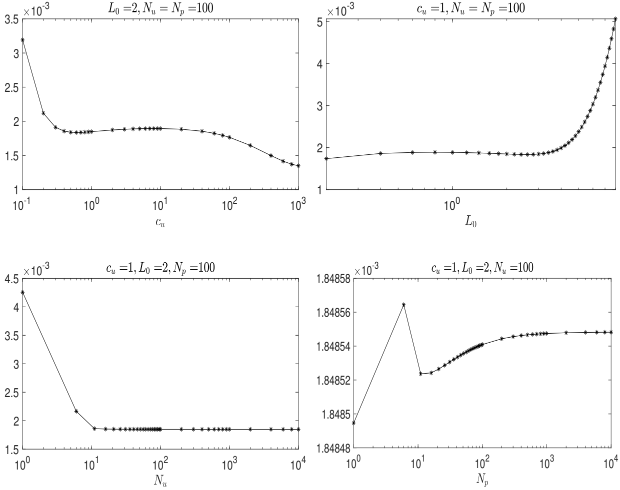


Figure: Parameter tests for a fixed h, from top to bottom: c_u, L_0, N_u, and N_p variations.

The square domain IV

Table: Errors and rates of convergence (in brackets) for the square domain test on different triangulations.

Triangulation	h	$\lVert \boldsymbol{e_u} \rVert / \lVert \boldsymbol{u} \rVert$	$\lVert \nabla \times \boldsymbol{e_u} \rVert / \lVert \nabla \times \boldsymbol{u} \rVert$
Uniform right-angled	0.3536	1.07e-01	3.63e-01
	0.1768	2.04e-02 (2.39)	1.72e-01 (1.08)
	0.0884	4.75e-03 (2.10)	8.62e-02 (1.00)
	0.0442	1.18e-03 (2.01)	4.32e-02 (1.00)
Criss-cross	0.2500	6.34e-02	1.45e-01
	0.1250	1.60e-02 (1.98)	7.05e-02 (1.04)
	0.0625	4.02e-03 (2.00)	3.50e-02 (1.01)
	0.0312	1.01e-03 (2.00)	1.75e-02 (1.00)
Powell-Sabin	0.1913	2.91e-02	9.86e-02
	0.0957	7.38e-03 (1.98)	4.80e-02 (1.04)
	0.0478	1.85e-03 (2.00)	2.38e-02 (1.01)
	0.0239	4.62e-04 (2.00)	1.19e-02 (1.00)

The square domain V

Table: Strong imposition of the boundary conditions using Powell-Sabin meshes on the square domain.

h	$\|\boldsymbol{e_u}\|/\|\boldsymbol{u}\|$	$\|\nabla \times \boldsymbol{e_u}\|/\|\nabla \times \boldsymbol{u}\|$
0.1913	2.90e-02	9.87e-02
0.0957	7.38e-03 (1.98)	4.80e-02 (1.04)
0.0478	1.85e-03 (2.00)	2.38e-02 (1.01)
0.0239	4.62e-04 (2.00)	1.19e-02 (1.00)

The L-shaped domain I

Now $\Omega = [-1,1]^2 \setminus \{[0,1] \times [-1,0]\}$. The manufactured solution is $\boldsymbol{u} = \nabla\psi$ where $\psi(r,\theta) = r^{2n/3}\sin(2n\theta/3)$, for different levels of smoothness depending on n.

Table: Errors and rates of convergence (in brackets) for the L-shaped domain test on criss-cross triangulations.

n	h	$\Vert\boldsymbol{e_u}\Vert/\Vert\boldsymbol{u}\Vert$	$\Vert\nabla\times\boldsymbol{e_u}\Vert/\Vert\nabla\boldsymbol{u}\Vert$
1	0.1250	2.61e-01	2.55e-01
	0.0625	1.58e-01 (0.72)	7.93e-02 (1.69)
	0.0312	9.38e-02 (0.76)	2.47e-02 (1.69)
	0.0156	5.66e-02 (0.73)	7.71e-03 (1.68)
2	0.1250	2.12e-02	4.62e-02
	0.0625	9.80e-03 (1.12)	1.29e-02 (1.84)
	0.0312	4.15e-03 (1.24)	3.40e-03 (1.92)
	0.0156	1.69e-03 (1.30)	8.72e-04 (1.96)
4	0.1250	3.09e-03	3.06e-03
	0.0625	8.33e-04 (1.89)	3.97e-04 (2.94)
	0.0312	2.12e-04 (1.98)	5.01e-05 (2.99)
	0.0156	5.31e-05 (1.99)	6.26e-06 (3.00)

The L-shaped domain II

Table: Errors and rates of convergence (in brackets) for the L-shaped domain test on Powell-Sabin triangulations.

n	h	$\Vert \boldsymbol{e_u} \Vert / \Vert \boldsymbol{u} \Vert$	$\Vert \nabla \times \boldsymbol{e_u} \Vert / \Vert \nabla \boldsymbol{u} \Vert$
1	0.0957	2.11e-01	1.60e-01
	0.0478	1.25e-01 (0.76)	4.95e-02 (1.69)
	0.0239	7.40e-02 (0.76)	1.54e-02 (1.68)
	0.0120	4.49e-02 (0.72)	4.83e-03 (1.68)
2	0.0957	1.63e-02	2.61e-02
	0.0478	6.94e-03 (1.23)	5.86e-03 (2.16)
	0.0239	2.81e-03 (1.30)	1.23e-03 (2.26)
	0.0120	1.12e-03 (1.33)	2.50e-04 (2.30)
4	0.0957	1.63e-03	1.45e-03
	0.0478	4.27e-04 (1.93)	1.85e-04 (2.97)
	0.0239	1.08e-04 (1.99)	2.29e-05 (3.01)
	0.0120	2.69e-05 (2.00)	2.84e-06 (3.02)

The L-shaped domain III

Table: A comparison of different ways to impose the boundary conditions on the L-shaped domain when $n = 1$ and $h = 0.0156$: Strong imposition (different strategies at the re-entrant corner) and Nitsche's method.

Strategy	$\|\boldsymbol{e_u}\|/\|\boldsymbol{u}\|$	$\|\nabla \times \boldsymbol{e_u}\|/\|\nabla \boldsymbol{u}\|$
$u_1 = u_2 = 0$	5.66e-02	7.65e-03
u_1, u_2 free	2.82e-02	1.27e-03
Bisector normal	2.82e-02	1.27e-03
Nitsche's method	5.66e-02	7.71e-03

The curved L-shape domain I

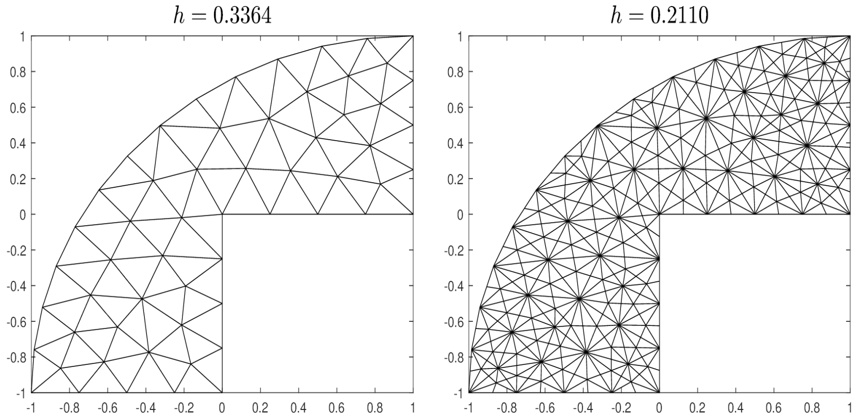


Figure: The regular unstructured (L) and Powell-Sabin type (R) meshing of the curved L-shape domain.

The curved L-shape domain II

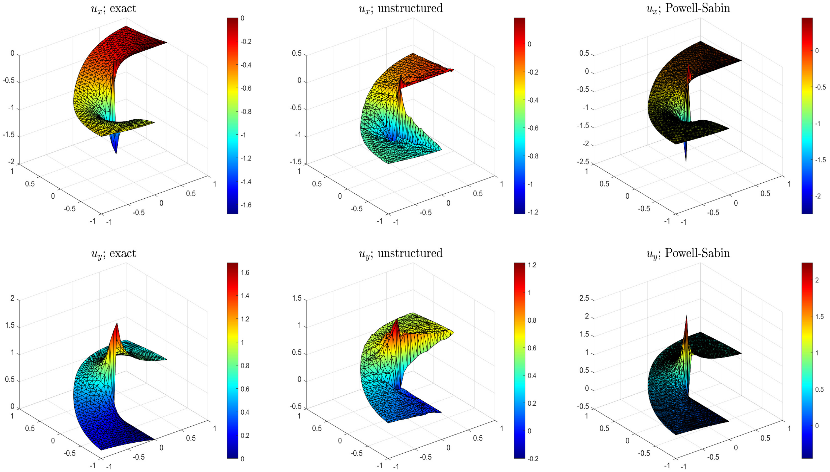


Figure: Exact, unstructured mesh, and Powell-Sabin mesh solution components.

The curved L-shape domain III

Table: Errors and rates of convergence (in brackets) for the curved L-shape domain test on Powell-Sabin triangulations.

n	h	$\|\boldsymbol{e_u}\|/\|\boldsymbol{u}\|$	$\|\nabla \times \boldsymbol{e_u}\|/\|\nabla \boldsymbol{u}\|$
1	0.4163	3.15e-01	3.31e-01
	0.2110	2.22e-01 (0.50)	1.12e-01 (1.56)
	0.1192	1.42e-01 (0.65)	4.31e-02 (1.38)
	0.0586	9.02e-02 (0.65)	1.37e-02 (1.66)
	0.0287	5.59e-02 (0.69)	4.20e-03 (1.70)
	0.0146	3.47e-02 (0.69)	1.36e-03 (1.63)
2	0.4163	4.49e-02	6.86e-02
	0.2110	2.66e-02 (0.76)	2.02e-02 (1.77)
	0.1192	1.19e-02 (1.15)	5.14e-03 (1.97)
	0.0586	5.33e-03 (1.16)	1.09e-03 (2.24)
	0.0287	2.13e-03 (1.32)	2.18e-04 (2.32)
	0.0146	8.07e-04 (1.40)	4.51e-05 (2.27)
4	0.4163	2.52e-02	2.00e-02
	0.2110	7.79e-03 (1.69)	3.08e-03 (2.70)
	0.1192	1.77e-03 (2.14)	5.16e-04 (2.58)
	0.0586	4.90e-04 (1.85)	9.96e-05 (2.37)
	0.0287	1.17e-04 (2.06)	2.22e-05 (2.17)
	0.0146	2.82e-05 (2.06)	5.44e-06 (2.03)

The cube domain

Let $\Omega = (0,1)^3$ and consider the manufactured solution:

$$\boldsymbol{u}(x,y,z) = (\sin(x)(\cos(y) - \cos(z)), \sin(y)(\cos(z) - \cos(x)), \sin(z)(\cos(x) - \cos(y)))$$

The results obtained are:

Table: Errors and rates of convergence (in brackets) for the cube domain.

h	$\|\boldsymbol{e_u}\| \ / \|\boldsymbol{u}\|$	$\|\nabla \times \boldsymbol{e_u}\| / \|\nabla \times \boldsymbol{u}\|$
0.8660	1.61e-01	3.44e-01
0.4330	3.74e-02 (2.10)	1.63e-01 (1.07)
0.2165	9.16e-03 (2.03)	8.01e-02 (1.03)
0.1083	2.29e-03 (2.00)	3.98e-02 (1.01)

Outline

Conclusions

Two FE element formulations have been considered:

- the Galerkin method when implemented with inf-sup stable elements,
- an augmented-stabilised method that permits the use of nodal interpolations of arbitrary order.

We have provided stability and optimal convergence results for both of the formulations. Numerical results have been used to:

- confirm the expected orders of convergence, both for smooth and for singular solutions,
- confirm the need of using meshes able to capture singular solutions,
- shown the effectiveness of Nitsche's method to approximate Dirichlet BCs.

Main references

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Submitted.

THANK YOU!

...and my best wishes to Kisko and Manolo!!!