Control of parabolic problems and block moment method

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Abstract linear control problem

$$\begin{cases} y'(t) + \mathcal{A}y(t) = \mathcal{B}u(t), & t \in (0,T), \\ y(0) = y_0. \end{cases}$$

- $-\mathcal{A}$ generates a C^0-semigroup on the Hilbert space $(X, \|\cdot\|)$,
- The space of controls is the Hilbert space $(U, \|\cdot\|_U)$.
- The control operator $\mathcal{B} : U \to D(\mathcal{A}^*)'$. Assume (for simplicity) that

$$\int_0^T \left\| \mathcal{B}^* e^{-t\mathcal{A}^*} z \right\|_U^2 \mathrm{d}t \leq C \|z\|^2, \qquad \forall z \in D(\mathcal{A}^*).$$

Notion of solution

Wellposedness theorem

Let $T > 0$. For any $y_0 \in X$ and any $u \in L^2(0,T;U)$, there exists a unique solution $y \in C^0([0,T],X)$ characterized by

$$\langle y(t), z\rangle - \left\langle y_0, e^{-t\mathcal{A}^*} z\right\rangle = \int_0^t \left\langle u(\tau), \mathcal{B}^* e^{-(t-\tau)\mathcal{A}^*} z\right\rangle_U \mathrm{d}\tau,$$

for any $t \in [0,T]$, and any $z \in X$.
Moreover, there exists $C > 0$ such that for any such y_0, u, the solution satisfies

$$\|y(t)\| \leq C \left(\|y_0\| + \|u\|_{L^2(0,T;U)}\right), \quad \forall t \in [0,T].$$

- **Question :** null controllability of a given y_0 at a given time $T > 0$:

$$\exists u \in L^2(0,T;U) \; ; \; y(T) = 0?$$

Typical examples

- Boundary control of coupled equations

$$\begin{cases} \partial_t y_1 - \Delta y_1 + y_2 = 0, & \text{in } (0,T) \times \Omega, \\ \partial_t y_2 + \big(- \Delta + c(x) \big) y_2 = 0, & \text{in } (0,T) \times \Omega, \\ {y_1}_{|\partial\Omega} = 0, \quad {y_2}_{|\partial\Omega} = \mathbf{1}_\Gamma u & \text{in } (0,T) \times \partial\Omega, \\ y_1(0,\cdot) = y_{1,0}, \quad y_2(0,\cdot) = y_{2,0}. \end{cases}$$

- Simultaneous controllability

$$\begin{cases} \partial_t y_1 - \Delta y_1 = \mathbf{1}_\omega u, & \text{in } (0,T) \times \Omega, \\ \partial_t y_2 + \big(- \Delta + c(x) \big) y_2 = \mathbf{1}_\omega u, & \text{in } (0,T) \times \Omega, \\ {y_1}_{|\partial\Omega} = {y_2}_{|\partial\Omega} = 0 & \text{in } (0,T) \times \partial\Omega, \\ y_1(0,\cdot) = y_{1,0}, \quad y_2(0,\cdot) = y_{2,0}. \end{cases}$$

1. Control of parabolic problems and moment problems
   - Moment problems and biorthogonal families
   - A limitation in the use of biorthogonal families
2. The block moment method for scalar controls
   - Setting
   - The block moment problem and its resolution
3. How does it relate to similar results for wave-like equations?
4. The block moment method for general control operators
5. Biorthogonal families in higher dimension
   - Setting and biorthogonal families
   - Ingredients of proof

Assumptions and moment problem

The setting

- Assume that the operator $\mathcal{A}^*$ admits a sequence of positive eigenvalues Λ.
- We denote by $(\phi_\lambda)_{\lambda\in\Lambda}$ the associated sequence of normalized eigenvectors and we assume that it forms a complete family in X.

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Definition of solutions: for all $\lambda \in \Lambda$,

$$\langle y(T), \phi_\lambda \rangle - \left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle = \int_0^T \left\langle u(t), e^{-\lambda(T-t)}\mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t.$$

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Complete family of eigenvectors $(\phi_\lambda)_{\lambda\in\Lambda}$:

$$y(T) = 0 \quad \Longleftrightarrow \quad \int_0^T \left\langle u(t), e^{-\lambda(T-t)}\mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda$$

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$$\Longleftrightarrow \quad \boxed{\int_0^T \left\langle v(t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda}$$

with $v := u(T - \cdot)$.

Reduction to a moment problem when $\dim U = 1$

- Scalar control ($\dim U = 1$) with observable eigenvectors ($\mathcal{B}^*\phi_\lambda \neq 0$)

$$
\begin{aligned}
y(T) = 0 \quad &\Longleftrightarrow \quad \int_0^T e^{-\lambda t} \left\langle v(t), \mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda \\
&\Longleftrightarrow \quad \mathcal{B}^*\phi_\lambda \int_0^T e^{-\lambda t} v(t)\mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \ \forall \lambda \in \Lambda \\
&\Longleftrightarrow \quad \boxed{\int_0^T e^{-\lambda t} v(t)\mathrm{d}t = -e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda} \right\rangle, \ \forall \lambda \in \Lambda}
\end{aligned}
$$

Resolution of the moment problem using a biorthogonal family

$$\text{Find } v \text{ such that } \int_0^T e^{-\lambda t} v(t)\mathrm{d}t = -e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}\right\rangle, \ \forall \lambda \in \Lambda$$

Biorthogonal family $(q_\lambda)_{\lambda\in\Lambda}$ to the exponentials associated with Λ in $L^2(0,T;\mathbb{R})$

$$\begin{cases} \displaystyle\int_0^T e^{-\mu t} q_\lambda(t)\mathrm{d}t = 0, \quad \forall \mu \in \Lambda\backslash\{\lambda\}, \\ \displaystyle\int_0^T e^{-\lambda t} q_\lambda(t)\mathrm{d}t = 1. \end{cases}$$

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Existence of such biorthogonal family $\overset{\text{Schwartz}}{\Longleftrightarrow} \displaystyle\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty.$

Applications to heat-like PDEs are restricted to the **1D case**.

Resolution of the moment problem using a biorthogonal family

$$\text{Find } v \text{ such that } \int_0^T e^{-\lambda t} v(t) \mathrm{d}t = -e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle, \ \forall \lambda \in \Lambda$$

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Existence of such biorthogonal family $\overset{\text{Schwartz}}{\Longleftrightarrow} \displaystyle\sum_{\lambda \in \Lambda} \frac{1}{\lambda} < +\infty.$

In this case,

$$u : t \in (0,T) \mapsto -\sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(T-t)$$

formally solves the moment problem.

Question: estimate $\mathcal{B}^* \phi_\lambda$ and $\|q_\lambda\|_{L^2(0,T;\mathbb{R})}$ to prove that the series converges in $L^2(0,T;\mathbb{R})$.

Some estimates on biorthogonal families

Under the gap condition ($|\lambda - \mu| > \rho, \quad \forall \lambda \neq \mu \in \Lambda$).

- H.O. Fattorini & D.L Russell (1974): $\|q_\lambda\|_{L^2(0,T;\mathbb{R})} \leq C_{\varepsilon,T} e^{\varepsilon \lambda}$.
  Uniform estimates with respect to Λ in a certain class.
- A. Benabdallah, F. Boyer, M. González Burgos & G. Olive (2014)
  Sharper estimates + dependency $/T$: $\|q_\lambda\|_{L^2(0,T;\mathbb{R})} \leq C e^{C/T} e^{C\sqrt{\lambda}}$.
- P. Cannarsa, P. Martinez & J. Vancostenoble (2020)
  Optimal estimates + dealing with asymptotic gap.

Under a weak gap condition (gap between blocks of bounded cardinality)

- N. Cîndea, S. Micu, I. Roventa & M.Tucsnak (2015)
  Union of two sequences with gap condition plus a non-condensation assumption
- A. Benabdallah, F. Boyer & M. M. (2020)
- M. González Burgos & L. Ouaili (2020)

Without any gap condition

- F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014)
  Condensation index of the sequence.
- D. Allonsius, F. Boyer & M. Morancey (2021)
  "Local" gap for each λ.

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + \exp(a\partial_{xx}) \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix}.$$

Eigenvectors of $-\partial_{xx}$: $-\partial_{xx}\varphi_k = k^2\varphi_k$. Thus,

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + e^{-ak^2}\ ;\ k \in \mathbb{N}^* \right\}$$

Complete family of associated eigenvectors of $\mathcal{A}^*$:

$$\phi_{k,1} = \begin{pmatrix} -e^{-ak^2} \\ 1 \end{pmatrix} \varphi_k, \qquad \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014):
there exists a biorthogonal family satisfying

$$\frac{1}{C_\varepsilon} e^{(a-\varepsilon)\lambda} \leq \|q_\lambda\|_{L^2(0,T;\mathbb{R})} \leq C_\varepsilon e^{(a+\varepsilon)\lambda}.$$

Limitation in the use of biorthogonal families

Formal solution of moment problem given by

$$u : t \in (0,T) \mapsto -\sum_{\lambda \in \Lambda} e^{-\lambda T} \left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda} \right\rangle q_\lambda(T-t)$$

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$\longrightarrow$ Direct application of moments method yields null controllability in time $T > a$.

Yet, we will see that the previous example is null controllable in any time $T > 0$...

What is missed in the direct application of the moment method?
Only information on $\|q_\lambda\|$: proof of normal convergence of the series in $L^2(0,T;\mathbb{R})$ which is not the most subtle convergence...

As $\lambda_{k,1} \approx \lambda_{k,2}$, it can be a good idea to consider the control u in the form

$$u : t \in (0,T) \mapsto -\sum_{k\geq 1}\left(\sum_{j=1}^{2} e^{-\lambda_{k,j}T}\left\langle y_0, \frac{\phi_{k,j}}{\mathcal{B}^*\phi_{k,j}}\right\rangle q_{k,j}(T-t)\right)$$

and estimate

$$\left\|\sum_{j=1}^{2} e^{-\lambda_{k,j}T}\left\langle y_0, \frac{\phi_{k,j}}{\mathcal{B}^*\phi_{k,j}}\right\rangle q_{k,j}(T-\cdot)\right\|_{L^2(0,T)}.$$

Assumptions

$\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions for the wellposedness.

- Scalar control $U = \mathbb{R}$.
- Eigenvalues of $\mathcal{A}^*$.
  - Λ: positive simple eigenvalues of $\mathcal{A}^*$ satisfying $\displaystyle\sum_{\lambda\in\Lambda} \frac{1}{\lambda} < +\infty$.
  - asymptotic behavior of the counting function: $N_\Lambda(r) := \mathrm{Card}\,\{\lambda \in \Lambda\,;\, \lambda \leq r\} \leq \kappa r^\theta$ with $\theta \in (0,1)$.
- $(\phi_\lambda)_{\lambda\in\Lambda}$ associated eigenvectors.
  - complete family of eigenvectors in X.
  - $\mathrm{Ker}(\mathcal{A}^* - \lambda) \cap \mathrm{Ker}\,\mathcal{B}^* = \{0\}$ for every $\lambda \in \mathbb{R}$.

Extra assumption :

- Weak gap condition: there exists $\rho > 0$ and $p \in \mathbb{N}^*$ such that

$$\mathrm{Card}\,(\Lambda \cap [\mu, \mu+\rho]) \leq p, \qquad \forall \mu \geq 0.$$

Groups of eigenvalues

Let $p \in \mathbb{N}^*$ and $\rho > 0$. The weak-gap condition ensures the existence of sets $(G_k)_{k\geq 1} \subset \mathcal{P}(\Lambda)$ such that

$$\Lambda = \bigcup_{k \geq 1} G_k, \qquad \sup(G_k) < \inf(G_{k+1}),$$

with the additional properties that for every $k \geq 1$,

$$g_k := \# G_k \leq p, \qquad \operatorname{dist}(G_k, G_{k+1}) \geq r, \qquad \operatorname{diam} G_k < \rho.$$

with $r = r_{p,\rho} > 0$.

- Labelling the eigenelements

$$G_k = \{\lambda_{k,1}, \dots, \lambda_{k,g_k}\} \qquad \text{with } \lambda_{k,1} < \dots < \lambda_{k,g_k},$$

$$\phi_{k,j} := \phi_{\lambda_{k,j}}, \qquad \forall k \geq 1, \ \forall 1 \leq j \leq g_k.$$

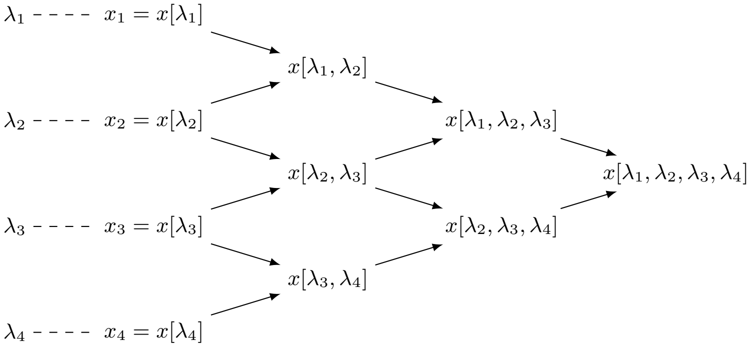
- The analysis is insensitive to the particular choice of such a grouping.

Divided differences in a given group G_k

- For any j, set $x[\lambda_j] := x_j$.
- Divided differences. For any $i \neq j$ we set

$$x[\lambda_i, \lambda_j] := \frac{x[\lambda_j] - x[\lambda_i]}{\lambda_j - \lambda_i} \in X.$$

and so on ... following the diagram



$$y(T)=0 \iff \int_0^T e^{-\lambda(T-t)}u(t)\mathrm{d}t = -e^{-\lambda T}\left\langle y_0, \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}\right\rangle, \ \forall \lambda\in\Lambda$$

$$\iff \int_0^T e^{-\lambda(T-t)}u(t)\mathrm{d}t = -e^{-\lambda T}\left\langle y_0, \psi_\lambda\right\rangle, \ \forall \lambda\in\Lambda$$

$$\text{where } \psi_\lambda := \frac{\phi_\lambda}{\mathcal{B}^*\phi_\lambda}.$$

Look for u in the form

$$u : t\in(0,T)\mapsto -\sum_{k\geq 1} v_k(T-t)$$

where

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j}t}v_k(t)\mathrm{d}t = e^{-\lambda_{k,j}T}\left\langle y_0,\psi_{k,j}\right\rangle, & \forall k\geq 1,\ \forall 1\leq j\leq g_k, \\ \displaystyle\int_0^T e^{-\lambda t}v_k(t)\mathrm{d}t = 0, & \forall\lambda\in\Lambda\backslash G_k. \end{cases}$$

The function v_k solves the moment problem inside the group G_k.

A. Benabdallah, F. Boyer & M. M. (2020)

Let $T \in (0, +\infty]$. For any $\varepsilon > 0$, there exists a constant $C > 0$ such that for any $k \geq 1$, for any $\omega_{k,1}, \dots, \omega_{k,g_k} \in \mathbb{R}$, there exists $v_k \in L^2(0,T;\mathbb{R})$ satisfying

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j} t} v_k(t)\mathrm{d}t = \omega_{k,j}, & \forall 1 \leq j \leq g_k, \\ \displaystyle\int_0^T e^{-\lambda t} v_k(t)\mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \end{cases}$$

and

$$\|v_k\|_{L^2(0,T;\mathbb{R})} \leq C e^{C/T^{\frac{\theta}{1-\theta}}} e^{C\lambda_{k,1}^{\theta}} \max_{1 \leq l \leq g_k} \left| \omega[\lambda_{k,1}, \dots, \lambda_{k,l}] \right|.$$

Moreover, up to the exponential factors, this last estimate is sharp.

Adaptation of H.O. Fattorini & D.L. Russell (1974) using the isomorphism of the Laplace transform and refined estimates using Paley-Wiener theorem (F. Boyer - M2 lecture notes (HAL))

- Block moment problem associated to null controllability

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j}t} v_k(t)\mathrm{d}t = e^{-\lambda_{k,j}T} \left\langle y_0, \psi_{k,j} \right\rangle, & \forall k \geq 1,\ \forall 1 \leq j \leq g_k, \\ \displaystyle\int_0^T e^{-\lambda t} v_k(t)\mathrm{d}t = 0, & \forall \lambda \in \Lambda \backslash G_k, \end{cases}$$

where $\psi_\lambda = \frac{\phi_\lambda}{\mathcal{B}^* \phi_\lambda}$. The solution satisfies

$$\|v_k\|_{L^2(0,T;\mathbb{R})} \leq C e^{C/T^{\frac{\theta}{1-\theta}}} e^{C\lambda_{k,1}^{\theta}} e^{-\lambda_{k,1}T} \max_{1 \leq l \leq g_k} \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\|.$$

- Sufficiently sharp estimates to characterize the minimal null control time as

$$T_0 = \limsup_{k \to \infty} \frac{\ln\left(\displaystyle\max_{1 \leq l \leq g_k} \|\psi[\lambda_{k,1}, \dots, \lambda_{k,l}]\| \right)}{\lambda_{k,1}}.$$

Comments

- Extension to complex eigenvalues in a sector of dominant real part.
- Uniform estimates: similar results for algebraically multiple eigenvalues (limit process λ, $\lambda + h$).
- Application
  - K. Bhandari & F. Boyer (2021): boundary control, from Robin to Dirichlet boundary conditions.
  - F. Boyer & G. Olive (2023): 2D coupled heat equations with different constant diffusion coefficient.

Back to the academic example

$$\mathcal{A}y = \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + \exp(a\partial_{xx}) \end{pmatrix} y, \qquad \mathcal{B} = \begin{pmatrix} 0 \\ \text{a nice scalar control operator} \end{pmatrix}.$$

$$\Lambda = \left\{ \lambda_{k,1} := k^2,\ \lambda_{k,2} := k^2 + e^{-ak^2} \ ;\ k \in \mathbb{N}^* \right\} \quad \Longrightarrow \quad \#G_k = 2$$

$$T_0 = \limsup_{k\to\infty} \frac{1}{\lambda_{k,1}} \ln \max \left\{ \frac{1}{|\mathcal{B}^*\phi_{k,1}|}, \frac{1}{|\mathcal{B}^*\phi_{k,2}|}, \frac{\left\| \frac{\phi_{k,2}}{\mathcal{B}^*\phi_{k,2}} - \frac{\phi_{k,1}}{\mathcal{B}^*\phi_{k,1}} \right\|}{\lambda_{k,2} - \lambda_{k,1}} \right\} = 0.$$

Indeed,

$$\phi_{k,1} = \begin{pmatrix} -e^{-ak^2} \\ 1 \end{pmatrix} \varphi_k, \qquad \phi_{k,2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi_k,$$

imply

$$\mathcal{B}^*\phi_{k,1} = \mathcal{B}^*\phi_{k,2} = \text{ nice} \qquad \text{and} \qquad \|\phi_{k,2} - \phi_{k,1}\| = e^{-ak^2} = |\lambda_{k,2} - \lambda_{k,1}|.$$

The condensation of eigenvectors compensates the condensation of eigenvalues.

A PDE example behaving as the academic example

$$\begin{cases} \partial_t y(t,x) + \begin{pmatrix} -\partial_{xx} & 1 \\ 0 & -\partial_{xx} + c(x) \end{pmatrix} y(t,x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & (t,x) \in (0,T) \times (0,1), \\ y(t,0) = \begin{pmatrix} 0 \\ u(t) \end{pmatrix}, \quad y(t,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad t \in (0,T), \\ y(0,x) = y_0(x), \qquad x \in (0,1), \end{cases}$$

For any $c \in L^2(0,1;\mathbb{R})$

- possible presence of algebraically double eigenvalues;
- possible strong condensation of eigenvalues;
- possible (finite number of) non observable modes.

There exists $Y_0 \subset (H^{-1}(0,1;\mathbb{R}))^2$ with finite codimension such that

- if $y_0 \notin Y_0$: not approximately controllable;
- if $y_0 \in Y_0$: null controllability in any time $T > 0$.

$$\begin{cases} \partial_{ss} w - \partial_{xx} w + w = 0, & (s,x) \in (0,S) \times (0,1), \\ w(s,0) = u(s), \quad w(s,1) = 0, \\ (w, w_s)(0,\cdot) = (w_0, w_1). \end{cases}$$

- Eigenvalues of $\mathcal{A}^*$: $\mu_k = \operatorname{sign}(k)\sqrt{k^2\pi^2+1}$ for $k \in \mathbb{Z}$.
- Gap-condition: $\gamma = \inf_{k\in\mathbb{Z}} |\mu_{k+1} - \mu_k| > 0$.
- Ingham's inequality

For any $S > \frac{2\pi}{\gamma}$,

$$\frac{1}{C} \sum_{k\in\mathbb{Z}} |a_k|^2 \leq \int_0^S \left| \sum_{k\in\mathbb{Z}} a_k e^{i\mu_k s} \right|^2 \mathrm{d}s \leq C \sum_{k\in\mathbb{Z}} |a_k|^2.$$

Ingham's inequality $\Longrightarrow$ observability inequality ($\Longleftrightarrow$ controllability).

With a weak-gap condition (simultaneous control of strings with different lengths): generalized Ingham-type inequality for divided differences of the time-exponentials inside the blocks:

$$\frac{1}{C}\sum_{k\in\mathbb{Z}}\sum_{j=1}^{g_k}|a_{k,j}|^2 \leq \int_0^S \left|\sum_{k\in\mathbb{Z}}\sum_{j=1}^{g_k} a_k e_{is}\left[\mu_{k,1},\ldots,\mu_{k,j}\right]\right|^2 \mathrm{d}s \leq C\sum_{k\in\mathbb{Z}}\sum_{j=1}^{g_k}|a_k|^2$$

where $e_\bullet : x\in\mathbb{R}\mapsto e^{\bullet x}$.

Recall that

$$e_{is}\left[\mu_{k,1},\mu_{k,2}\right] = \frac{e^{i\mu_{k,2}s}-e^{i\mu_{k,1}s}}{\mu_{k,2}-\mu_{k,1}}.$$

See, for instance, V. Komornik & P. Loreti (2002).

Back to the parabolic world: $G_k = \{\lambda_{k,1}, \lambda_{k,2}\}$

Biorthogonal family to the divided differences of time exponentials inside the block:

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1}(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k, \end{cases} \qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1,2}(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1,2}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1,2}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k. \end{cases}$$

Back to the parabolic world: $G_k = \{\lambda_{k,1}, \lambda_{k,2}\}$

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1}(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k, \end{cases} \qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1,2}(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1,2}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1,2}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k. \end{cases}$$

- Block resolution $\Longrightarrow$ biorthogonal family to the divided differences

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T e^{-\lambda_{k,2} t} q_{k,1}(t)\mathrm{d}t = 1, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k, \end{cases} \qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1,2}(t)\mathrm{d}t = 0, \\ \displaystyle\int_0^T e^{-\lambda_{k,2} t} q_{k,1,2}(t)\mathrm{d}t = \lambda_{k,2} - \lambda_{k,1}, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1,2}(t)\mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k. \end{cases}$$

Back to the parabolic world: $G_k = \{\lambda_{k,1}, \lambda_{k,2}\}$

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1}(t) \mathrm{d}t = 1, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1}(t) \mathrm{d}t = 0, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1}(t) \mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k, \end{cases} \qquad \begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,1} t} q_{k,1,2}(t) \mathrm{d}t = 0, \\ \displaystyle\int_0^T \frac{e^{-\lambda_{k,2} t} - e^{-\lambda_{k,1} t}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1,2}(t) \mathrm{d}t = 1, \\ \displaystyle\int_0^T e^{-\lambda t} q_{k,1,2}(t) \mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k. \end{cases}$$

- Block resolution $\Longrightarrow$ biorthogonal family to the divided differences
- Block resolution $\Longleftarrow$ biorthogonal family to the divided differences
  Let
  $$v_k = \omega_{k,1} q_{k,1} + \frac{\omega_{k,2} - \omega_{k,1}}{\lambda_{k,2} - \lambda_{k,1}} q_{k,1,2}.$$
  Then,

$$\begin{cases} \displaystyle\int_0^T e^{-\lambda_{k,j} t} v_k(t) \mathrm{d}t = \omega_{k,j}, \qquad \forall j \in \{1, 2\}, \\ \displaystyle\int_0^T e^{-\lambda t} v_k(t) \mathrm{d}t = 0, \quad \lambda \in \Lambda \backslash G_k. \end{cases}$$

M. Mehrenberger, M. M. ($\geq$ 2025)

Solvability of block moment problems at cost

$$\|v_k\|_{L^2(0,T)} \leq \mathfrak{C}(T, G_k) \times \sum_{j=1}^{g_k} \left|\omega\left[\lambda_{k,1}, \ldots, \lambda_{k,j}\right]\right|, \quad \forall k \geq 1,$$

$\Longleftrightarrow$

Existence of a biorthogonal family $\left(q_{\ell,m}\right)_{\ell \geq 1, 1 \leq m \leq g_\ell}$ to the divided differences in the blocks of the time exponentials i.e. $\forall k, \ell \geq 1$, $\forall j : 1 \leq j \leq g_k$, $\forall m : 1 \leq m \leq g_\ell$,

$$\int_0^T e_{-t}\left[\lambda_{k,1}, \ldots, \lambda_{k,j}\right] q_{\ell,m}(t) \mathrm{d}t = \delta_{k\ell}\delta_{jm}$$

with

$$\|q_{\ell,m}\|_{L^2(0,T)} \leq \mathfrak{C}(T, G_\ell).$$

Alternative proof of resolution of block moment problems

Follows a remark from C. Laurent & M. Léautaud (2023).

Assume that $\sqrt{\Lambda}$ satisfies a weak-gap condition.

- Generalized Ingham-type inequality for the divided differences of the complex time exponentials associated with $\sqrt{\Lambda}$;
- existence of a bounded biorthogonal family to this family;
- application of the transmutation transformation from S. Ervedoza & E. Zuazua (2011) to these biorthogonal elements;
- careful estimation of the divided differences gives

M. Mehrenberger, M. M. ($\geq$ 2025)

$$\int_0^T e_{-t}\left[\lambda_{k,1},\dots,\lambda_{k,j}\right] q_{\ell,m}(t)\mathrm{d}t = \delta_{k\ell}\delta_{jm}$$

with

$$\|q_{\ell,m}\|_{L^2(0,T)} \leq Ce^{C/T}e^{C\sqrt{\lambda_{\ell,1}}}, \qquad \forall \ell \geq 1,\ \forall m : 1 \leq m \leq g_\ell.$$

And thus resolution of the block moment problems but under the (more restrictive) condition that $\sqrt{\Lambda}$ satisfies a weak-gap condition

F. Boyer & M. M. (2023)

Resolution of block moment problems with almost sharp estimates and computation of the minimal null control time under the same assumptions as in the scalar case except that '$\dim U = 1$' is replaced by 'U a Hilbert space'.

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For instance, let $A\bullet = -\partial_x\big(\gamma(x)\partial_x \bullet\big) + c(x)\bullet$ and consider

$$\begin{cases} \partial_t y + \begin{pmatrix} A & 1 \\ 0 & dA \end{pmatrix} y = 0, & t \in (0,T), x \in (0,1), \\ y(t,0) = \begin{pmatrix} u_0(t) \\ u_0(t) \end{pmatrix}, & y(t,1) = \begin{pmatrix} 0 \\ u_1(t) \end{pmatrix}. \end{cases}$$

- F. Ammar Khodja, A. Benabdallah, M. González Burgos & L. de Teresa (2014). Assume that $A = -\partial_{xx}$, $u_0 \equiv 0$ and $\sqrt{d} \notin \mathbb{Q}$. Then,

$$T_0\left(H^{-1}(0,1;\mathbb{R})^2\right) = \limsup_{k\to+\infty} \frac{-\ln|\lambda_{k+1}-\lambda_k|}{\lambda_k},$$

and for any $\tau \in [0,+\infty]$, there exists $d \in (0,+\infty)$ such that $T_0 = \tau$.

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- F. Boyer & M. M. (2023).
  Using both controls u_0 and u_1, for any $d > 0$, there exists $Y_0 \subset (H^{-1}(0,1;\mathbb{R}))^2$ with finite codimension such that
  - if $y_0 \notin Y_0$: not approximately controllable;
  - if $y_0 \in Y_0$: null controllability in any time $T > 0$.

Space varying zero order coupling term

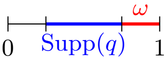
F. Boyer & M. M. (2025)

General expression of the minimal null control time for

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega(x) u(t,x) \end{pmatrix}, \quad t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = 0. \end{cases} \qquad (S_q)$$

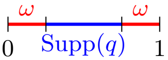
For example, with $A = -\partial_{xx}$ and $q(x) = \left(x - \frac{1}{2}\right) \mathbf{1}_{\left(\frac{1}{4}, \frac{3}{4}\right)}(x)$:

- F. Boyer & G. Olive (2014). If



  then the problem is not approximately controllable (for any time $T > 0$).

- If



  then $T_0\left(L^2(0,1;\mathbb{R})^2\right) = 0$.

F. Boyer & M. M. (2025)

General expression of the minimal null control time for

$$\begin{cases} \partial_t y + \begin{pmatrix} A & q(x) \\ 0 & A \end{pmatrix} y = \begin{pmatrix} 0 \\ \mathbf{1}_\omega(x) u(t,x) \end{pmatrix}, & t \in (0,T), x \in (0,1), \\ y(t,0) = y(t,1) = 0. \end{cases} \tag{S_q}$$

For example, with $A = -\partial_{xx}$, for any $\tau \in [0, +\infty]$, there exists $q, \tilde{q} \in L^\infty(0,1;\mathbb{R})$ such that

- systems (S_q) and $(S_{\tilde{q}})$ are null controllable in any time $T > 0$;
- the minimal time for simultaneous null controllability of systems (S_q) and $(S_{\tilde{q}})$ is τ.

Chiclana de la Frontera

It looks like this

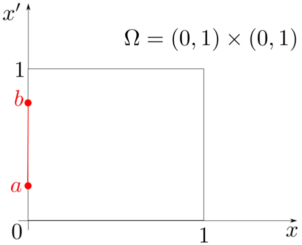
So you may imagine

Actually it rather looks like this

An example

Simultaneous controllability on $\Omega = (0,1) \times (0,1)$.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + c(x) \end{pmatrix} y = 0, \\ y_{|\partial\Omega} = \begin{pmatrix} \mathbf{1}_\Gamma u \\ \mathbf{1}_\Gamma u \end{pmatrix}. \end{cases}$$



The function c satisfies $\partial_{x'} c = 0$. $\Gamma = \{0\} \times (a, b)$.
Eigenelements: $\big(-\partial_{xx} + c(x) \big) \varphi_k^c(x) = \lambda_k^c \varphi_k^c(x)$.

Simultaneous controllability on $\Omega = (0,1) \times (0,1)$.

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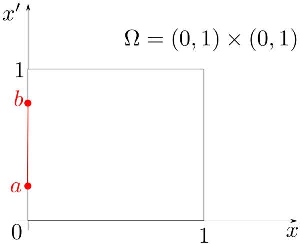

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Eigenelements: $\big(-\partial_{xx} + c(x)\big)\varphi_k^c(x) = \lambda_k^c \varphi_k^c(x)$.

- Eigenvalues of $\mathcal{A}^*$: Assume $\lambda_k^c \neq j^2\pi^2$, $\forall k, j \geq 1$.

$$\Lambda = \left\{k^2\pi^2 + m^2\pi^2 \;;\; k, m \geq 1\right\} \cup \left\{\lambda_k^c + m^2\pi^2 \;;\; k, m \geq 1\right\}.$$

- L. Ouaili (2019). 1D setting: minimal null control time (Dirichlet boundary condition at $x = 0$) given by the condensation index of the eigenvalues

$$T_0(c) = \limsup_{k \to +\infty} \frac{-\ln |k^2\pi^2 - \lambda_k^c|}{k^2\pi^2}.$$

- 2D setting: same minimal time with $\Gamma = \{0\} \times (0,1)$. But $\Gamma = \{0\} \times (a,b)$??

- Back to the moment problem

$$y(T) = 0 \quad \iff \quad \int_0^T \left\langle u(T-t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda \right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda \right\rangle, \, \forall \lambda \in \Lambda.$$

The multi-D moment problem

- Back to the moment problem

$$y(T)=0 \iff \int_0^T \left\langle u(T-t), e^{-\lambda t}\mathcal{B}^*\phi_\lambda\right\rangle_U \mathrm{d}t = -\left\langle y_0, e^{-\lambda T}\phi_\lambda\right\rangle, \ \forall\lambda\in\Lambda.$$

- Eigenvalues $\lambda_{k,m}^0 = k^2\pi^2 + m^2\pi^2$ and $\lambda_{k,m}^c = \lambda_k^c + m^2\pi^2$ with eigenvectors

$$(x,x') \mapsto \begin{pmatrix} \varphi_k^0(x)\sin(m\pi x') \\ 0 \end{pmatrix} \quad \text{and} \quad (x,x') \mapsto \begin{pmatrix} 0 \\ \varphi_k^c(x)\sin(m\pi x') \end{pmatrix}.$$

- Moment problem: find $v \in L^2((0,T)\times(a,b))$ such that for all $k,m \geq 1$,

$$\begin{cases} (\varphi_k^0)'(0)\displaystyle\int_0^T\int_a^b e^{-\lambda_{k,m}^0 t}\sin(m\pi x')v(t,x')\mathrm{d}x'\mathrm{d}t = -e^{-\lambda_{k,m}T}\left\langle y_0, \phi_{k,m}^0\right\rangle, \\ (\varphi_k^c)'(0)\displaystyle\int_0^T\int_a^b e^{-\lambda_{k,m}^c t}\sin(m\pi x')v(t,x')\mathrm{d}x'\mathrm{d}t = -e^{-\lambda_{k,m}T}\left\langle y_0, \phi_{k,m}^c\right\rangle. \end{cases}$$

The multi-D biorthogonal family

$$\begin{cases} (\varphi_k^0)'(0) \displaystyle\int_0^T \int_a^b e^{-\lambda_{k,m}^0 t} \sin(m\pi x') v(t,x') \mathrm{d}x' \mathrm{d}t = -e^{-\lambda_{k,m} T} \left\langle y_0, \phi_{k,m}^0 \right\rangle, \\ (\varphi_k^c)'(0) \displaystyle\int_0^T \int_a^b e^{-\lambda_{k,m}^c t} \sin(m\pi x') v(t,x') \mathrm{d}x' \mathrm{d}t = -e^{-\lambda_{k,m} T} \left\langle y_0, \phi_{k,m}^c \right\rangle. \end{cases}$$

- Look for a biorthogonal family in $L^2((0,T) \times (a,b))$ to $\{F_{k,m}^p \,;\, p \in \{0,c\}, k,m \geq 1\}$ with

$$F_{k,m}^p : (t,x') \mapsto e^{-\lambda_{k,m}^p t} \sin(m\pi x').$$

The multi-D biorthogonal family

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F. Ammar Khodja, A. Benabdallah, M. González Burgos, M. M. & L. de Teresa ($\geq$ 2025)

Construction of such biorthogonal family for any $T > 0$ with estimate

$$\|Q_{k,m}^p\|_{L^2((0,T)\times(a,b))} \leq C e^{C/T} e^{C\sqrt{\lambda_{k,m}^p}} \frac{1}{|\lambda_k^c - k^2\pi^2|}.$$

$\Longrightarrow$ Same minimal null control time as in the $1D$ setting.

First step: a nice biorthogonal family in $L^2((0,T)\times(0,1))$

- As $\lambda^p_{k,m} = \lambda^p_k + m^2\pi^2$, for any **fixed** $m \geq 1$, biorthogonal family $\left(q^p_{k,m}\right)$ in $L^2(0,T;\mathbb{R})$ to
$$t \in (0,T) \mapsto e^{-\lambda^p_{k,m} t}, \quad k \geq 1,$$
with estimate
$$\|q^p_{k,m}\| \leq C e^{C/T} e^{C\sqrt{\lambda_{k,m}}} \frac{1}{|\lambda^c_k - k^2\pi^2|}, \qquad \forall k,m \geq 1, p \in \{0,c\}.$$

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- Orthogonality in $L^2((0,1);\mathbb{R})$ of $(\sin(m\pi\cdot))_{m\geq 1}$ implies that

$$Q^p_{k,m} : (t,x') \mapsto q^p_{k,m}(t)\sin(m\pi x')$$

forms a biorthogonal family in $L^2((0,T)\times(0,1))$ to

$$F^p_{k,m} : (t,x') \mapsto e^{-\lambda^p_{k,m} t}\sin(m\pi x'), \qquad \forall k,m \geq 1$$

with estimate

$$\|Q^p_{k,m}\|_{L^2((0,T)\times(0,1))} \leq C e^{C/T} e^{C\sqrt{\lambda^p_{k,m}}} \frac{1}{|\lambda^c_k - k^2\pi^2|}, \qquad \forall k,m \geq 1, p \in \{0,c\}.$$

Same construction as F. Boyer & G. Olive (2023).

Second step: the restriction operator from $(0,1)$ to (a,b)

- Prove that the restriction in space operator

$$\begin{aligned} \mathcal{R} : \overline{\mathrm{Span}\{F^p_{k,m}\}}^{L^2_\rho((0,T)\times(0,1))} &\to \overline{\mathrm{Span}\{F^p_{k,m}\}}^{L^2((0,T)\times(a,b))} \\ F &\mapsto F_{|(a,b)} \end{aligned}$$

  is an isomorphism.

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  is an isomorphism.

- Follows from

$$\int_0^T \int_0^1 \rho(t) \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t \leq C \int_0^T \int_a^b \left|P_N(t,x')\right|^2 \mathrm{d}x'\mathrm{d}t$$

for any

$$P_N(t,x') = \sum_{k=1}^N \sum_{m=1}^N a^0_{k,m} e^{-\lambda^0_{k,m} t} \sin(m\pi x') + a^c_{k,m} e^{-\lambda^c_{k,m} t} \sin(m\pi x').$$

$$P_N(t,x') = \sum_{k=1}^{N} \sum_{m=1}^{N} \left(a_{k,m}^{0} e^{-\lambda_{k,m}^{0} t} + a_{k,m}^{c} e^{-\lambda_{k,m}^{c} t} \right) \sin(m\pi x')$$

- 1D spectral inequality in the variable x'

$$\int_0^1 \left| \sum_{m\leq\lambda} A_m \sin(m\pi x') \right|^2 \mathrm{d}x' \leq e^{\beta\lambda} \int_a^b \left| \sum_{m\leq\lambda} A_m \sin(m\pi x') \right|^2 \mathrm{d}x'$$

with a frequency cut depending on t (inspired by L. Miller (2010)).

$$P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N}\left(a_{k,m}^0 e^{-\lambda_{k,m}^0 t} + a_{k,m}^c e^{-\lambda_{k,m}^c t}\right)\sin(m\pi x')$$

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  with a frequency cut depending on t (inspired by L. Miller (2010)).
- Let $t\in(0,T)$ and $m\geq 1$ be fixed. Let $q_{k,m}^t$ be the solution of the block moment problem

$$\begin{cases} \displaystyle\int_0^T q_{k,m}^t(s)e^{-\lambda_{k,m}^0 s}\mathrm{d}s = e^{-\lambda_{k,m}^0 t}, \quad \int_0^T q_{k,m}^t(s)e^{-\lambda_{k,m}^c s}\mathrm{d}s = e^{-\lambda_{k,m}^c t}, \\ \displaystyle\int_0^T q_{k,m}^t(s)e^{-\lambda_{j,m}^p s}\mathrm{d}s = 0, \qquad \forall j\neq k,\ p\in\{0,c\}. \end{cases}$$

$$P_N(t,x') = \sum_{k=1}^{N}\sum_{m=1}^{N}\left(a^0_{k,m}e^{-\lambda^0_{k,m}t} + a^c_{k,m}e^{-\lambda^c_{k,m}t}\right)\sin(m\pi x')$$

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with a frequency cut depending on t (inspired by L. Miller (2010)).

- Let $t \in (0,T)$ and $m \geq 1$ be fixed. Let $q^t_{k,m}$ be the solution of the block moment problem

$$\begin{cases} \displaystyle\int_0^T q^t_{k,m}(s)e^{-\lambda^0_{k,m}s}\mathrm{d}s = e^{-\lambda^0_{k,m}t}, \quad \int_0^T q^t_{k,m}(s)e^{-\lambda^c_{k,m}s}\mathrm{d}s = e^{-\lambda^c_{k,m}t}, \\ \displaystyle\int_0^T q^t_{k,m}(s)e^{-\lambda^p_{j,m}s}\mathrm{d}s = 0, \qquad \forall j \neq k,\ p\in\{0,c\}. \end{cases}$$

Then,

$$\left\langle q^t_{k,m}\sin(m\pi\cdot), P_N\right\rangle_{L^2((0,T)\times(0,1))} = a^0_{k,m}e^{-\lambda^0_{k,m}t} + a^c_{k,m}e^{-\lambda^c_{k,m}t}$$

and

$$\|q^t_{k,m}\|_{L^2(0,T;\mathbb{R})} \leq Ce^{C/T}e^{C\sqrt{\lambda^0_{k,m}}}e^{-\lambda^0_{k,m}t}$$

Another example

Simultaneous controllability on $\Omega = (0,1) \times (0,1)$.

$$\begin{cases} \partial_t y + \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + c(x) \end{pmatrix} y = \begin{pmatrix} \mathbf{1}_{\omega \times (a,b)} u \\ \mathbf{1}_{\omega \times (a,b)} u \end{pmatrix}, \\ y_{|\partial\Omega} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

The function c satisfies $\partial_{x'} c = 0$.

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Construction of a suitable biorthogonal family with estimate

$$\|Q^p_{k,m}\|^2_{L^2((0,T)\times\omega\times(a,b))} \leq C e^{C/T} e^{C\sqrt{\lambda^p_{k,m}}} \frac{1}{\det \mathcal{G}_k + |\lambda^c_k - k^2\pi^2|^2}$$

where

$$\mathcal{G}_k = \mathrm{Gram}_{L^2(\omega)} \left(\varphi^0_k, \varphi^c_k\right).$$

$\Longrightarrow$ Minimal null control time if both eigenvalues and eigenvectors on ω condensate.

A general result

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- Cylindrical geometry and tensorized operators
- $\Lambda = \left\{ \lambda_k + \mu_m \; ; \; k, m \geq 1 \right\}$
- On the direction associated with λ_k: nice 1D assumptions (to solve block moment problems) on the eigenvalues.
- On the direction associated with μ_m: asymptotic of μ_m + Riesz-basis property for the eigenvectors + spectral inequality for the eigenvectors.

$\Longrightarrow$ construction and estimate of a space-time biorthogonal family for any time $T > 0$.

Conclusion and perspectives

Conclusion:

The block resolution of moment problems

- gives sharper results than the use of biorthogonal families ;
- allows to characterize the minimal null control time (of a given initial condition) for many parabolic-type one dimensional control problems for any admissible control operators ;
- is the parabolic equivalent of generalized Ingham-type results ;
- is a key tool to construct and estimate space-time biorthogonal families in higher dimension for tensorized problems.

Perspectives:

- The problem for non tensorized geometries or operators remains completely open...

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Thank you for your attention and feliz cumpleaños