The minimal control time for the exact controllability by internal controls of 1D linear hyperbolic systems

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(joint work with Long Hu)

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Outline of the talk

I. Framework

II. Boundary controllability : brief review

III. Proof of the main result

System description

Equations

$$\frac{\partial y}{\partial t}(t,x) + \Lambda(x)\frac{\partial y}{\partial x}(t,x) = M(x)y(t,x) + u(t,x), \quad (t,x) \in R_T,$$

- $R_T = (0,T) \times (0,1)$ and $y : R_T \to \mathbb{R}^n$ is the state.
- $\Lambda \in C^{0,1}([0,1])^{n\times n}$ is diagonal, with

$$\lambda_1 < \cdots < \lambda_m < 0 < \lambda_{m+1} < \cdots < \lambda_{m+p}.$$

- $M \in L^\infty(0,1)^{n\times n}$ is the **internal coupling matrix.**
- $u : R_T \to \mathbb{R}^n$ is the **control.** Constraint : $\operatorname{supp} u \subset (0,T) \times \omega$ with $\omega \subset (0,1)$ open, fixed.

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Denoting by $y = \begin{pmatrix} y_- \\ y_+ \end{pmatrix} \in \mathbb{R}^{m+p}$,

Boundary conditions

$$y_-(t,1) = Q_1 y_+(t,1), \quad y_+(t,0) = Q_0 y_-(t,0).$$

- $Q_1 \in \mathbb{R}^{m\times p}$ and $Q_0 \in \mathbb{R}^{p\times m}$ are the **boundary coupling matrices.**

Minimal control time

Well-posedness : $\forall y^0 \in L^2$, $\forall u \in L^2$, $\quad \exists! y \in C^0([0,T]; L^2(0,1)^n)$.

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Remark : **(EC)** in time $T_1 \quad \Longrightarrow \quad$ **(EC)** in time $T_2 \geq T_1$.

Definition

Minimal time for **(EC)** :

$$T_{\inf} = \inf \{ T > 0, \quad \text{System is } \textbf{(EC)} \text{ in time } T \}, \qquad (\in [0, +\infty]).$$

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- $T < T_{\inf} \quad \Longrightarrow \quad$ System is not **(EC)** in time T.

Goal

$$T_{\inf} = ??? \qquad (M, Q_1, Q_0 \text{ are fixed}).$$

Literature

Very few results, and only when $\omega = (a, b)$ is an **interval :**

Theorem ([**Zhuang, Li & Rao** (2016), DCDS] – also for quaslinear systems, [**Li, Lu & Qu** (2024), COCV])

Assume Q_1, Q_0 are invertible. Then, ***(EC)*** *in any time $T > (T_m + T_{m+1}) \times \max\{a, 1 - b\}$.* [a]

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Related result :

Theorem ([Alabau-Boussouira, Coron & Olive (2017), SICON])

Consider 2×2 underactuated quasilinear system with ***periodic boundary conditions :***

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Lambda(y)\dfrac{\partial y}{\partial x} = M(y) + \begin{pmatrix} u(t,x) \\ 0 \end{pmatrix}, \\ y(t,0) = y(t,1), \\ y(0,x) = y^0(x). \end{cases}$$

If $(\partial M_2/\partial y_1)(0,0) \neq 0$, then ***(EC)*** *in any time $T > \max\{T_1, T_2\} \times (1 - |b-a|)$ (locally near $y = 0$, for regular enough y^0, etc.).*

Main Result

Theorem ([Hu & Olive (2024), cocv])

- $T_{\inf} < +\infty \quad \Longrightarrow \quad Q_1, Q_0$ *are invertible (in part., $m = p$).*

- *If Q_1, Q_0 are invertible,*

$$T_{\inf} = \max_{I \in \mathcal{C}} T^{bc}_{\inf}(I),$$

where :

  - $\mathcal{C} = \{$*connected components of* $\overline{\omega}^c\}$.
  - $T^{bc}_{\inf}(I) =$ *minimal control time, on the interval I, **by boundary controls** (explicit ! See after).*

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- Strategy of proof : inspired by [**Ammar-Khodja, Benabdallah, González-Burgos & de Teresa** (2011), MCRF] on the implication

"boundary controllability $\Longrightarrow$ internal controllability"

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- Related results for 1D **parabolic systems** : [**Boyer & Olive** (2014), MCRF] and [**Boyer & Morancey** (2025), AMBP].

Examples

We assume that Q_1, Q_0 are invertible.

$$\omega = (a,b) \quad \Longrightarrow \quad T_{\inf} = \max\left\{ T_{\inf}^{bc}(0,a), \quad T_{\inf}^{bc}(b,1) \right\}.$$

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Lambda(x)\dfrac{\partial y}{\partial x}(t,x) = M(x)y(t,x), \\ y_-(t,a) = v_-(t) \quad y_+(t,0) = Q_0 y_-(t,0), \\ y(0,x) = y^0(x). \end{cases}$$

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$$\omega = (0,c) \cup (d,1) \quad \Longrightarrow \quad T_{\inf} = T_{\inf}^{bc}(c,d).$$

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We need to know $T_{\inf}^{bc}$ for **one-sided** and **two-sided** controllability.

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Control of a single equation

The transport equation :

$$\begin{cases} \dfrac{\partial y}{\partial t} + \lambda(x)\dfrac{\partial y}{\partial x} = 0, \\ y(t,1) = v(t) \text{ if } \lambda < 0, \quad (y(t,0) = v(t) \text{ if } \lambda > 0), \\ y(0,x) = y^0(x). \end{cases} \tag{TE}$$

We have

$$T_{\mathrm{inf}}^{bc} = \int_0^1 \frac{1}{|\lambda(\xi)|}\, d\xi.$$

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Coming back to systems,

Definition

$T_k = T_{\inf}^{bc}$ of (TE) with speed λ_k.

$$\lambda_1 < \cdots < \lambda_m \quad < 0 < \quad \lambda_{m+1} < \cdots < \lambda_{m+p},$$

$$\text{implies} \quad T_1 \leq \ldots \leq T_m, \qquad T_{m+1} \geq \ldots \geq T_{m+p}.$$

Two-sided boundary controllability

We consider

$$\begin{cases} \dfrac{\partial y}{\partial t}(t,x) + \Lambda(x)\dfrac{\partial y}{\partial x}(t,x) = M(x)y(t,x), \\ y_-(t,1) = v_-(t) \quad y_+(t,0) = v_+(t), \\ y(0,x) = y^0(x). \end{cases}$$

Theorem

$T_{\inf}^{bc} = \max\{T_m, \quad T_{m+1}\}$.

Upper bound $\leq$: [**Li & Rao** (2003), SICON] ("Constructive method").
Lower bound $\geq$: [**Hu & Olive** (2021), COCV] (Backstepping method).

One-sided boundary controllability

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Theorem ([Hu & Olive (2021), JMPA])

- $T_{\inf} < +\infty \implies \mathrm{rank}\, Q = p.$
- *If* $\mathrm{rank}\, Q = p$,

$$T_{\inf}^{bc} = \max_{1\leq k\leq p} \{T_{m+k} + T_{c_k}, \quad T_m\},$$

where c_k are indices from the LCU-decomposition of Q.

Proof by compactness-uniqueness ($T_{\inf}^{bc}$ is the same as for $M = 0$).

What are the c_k ?

Definition ([**Hu & Olive** (2022), JDE])

Q is in canonical form if there is at most one nonzero entry on each row and column, and $= 1$.

We denote by (r_k, c_k), $(1 \leq k \leq \rho)$ the corresponding indices, with $r_1 < \cdots < r_\rho$.

Examples :

$$\begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 & 0 \end{pmatrix}.$$

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The Gaussian elimination gives :

Proposition

There exist $L \in \mathbb{R}^{p\times p}$ (lower triang., $\operatorname{diag} L = 1$) and $U \in \mathbb{R}^{m\times m}$ (upper triang., invert.) :

$$LQU := Q^c \text{ is in canonical form.}$$

Moreover, Q^c is unique. We set $(r_k, c_k)(Q) = (r_k, c_k)(Q^c)$.

See also [**Dopico, Johnson & Molera** (2006), LAA].

Remark : $\operatorname{rank} Q = p$ implies $r_k = k$.

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Why should Q_1, Q_0 be invertible ?

The transport equation :

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is **never (EC).**

Remark : (TE) is approximately/null controllable in any time !

Proof for $\omega = (0, 1)$

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The proof is easy and well-known : we define

$$\bar{y} = \eta(t)y^f + (1 - \eta(t))y^b, \tag{1}$$

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Then, we simply take as control

$$\begin{aligned} \bar{u} &= \frac{\partial \bar{y}}{\partial t} + \Lambda(x) \frac{\partial \bar{y}}{\partial x} - M(x) \bar{y} \\ &= \eta'(t)(y^f - y^b). \end{aligned}$$

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Lemma

*There exists $\omega_0 \subset\subset \omega$, very close to ω (thus, **not small**), such that :*

- $\overline{\omega_0}^c = I_1 \cup I_2 \cup \ldots \cup I_N$ *(disjoint open intervals).*
- $T > T_{\inf}^{bc}(I_k)$ *for all k.*

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Then, as in [**Ammar-Khodja, Benabdallah, González-Burgos & de Teresa** (2011), MCRF],

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- In each $[0, T] \times I_k$: y^* is the controlled solution from the boundary.

The conclusion is as before :

$$\begin{aligned} u &= \frac{\partial y}{\partial t} + \Lambda(x)\frac{\partial y}{\partial x} - M(x)y \\ &= \xi'(x)\Lambda(x)(y^* - \bar{y}) + (1 - \xi(x))\bar{u}, \quad (\operatorname{supp} u \subset (0, T) \times \overline{\omega_1}). \end{aligned}$$

Open problems

- Underactuated systems : [1]

$$\frac{\partial y}{\partial t}(t,x) + \Lambda(x)\frac{\partial y}{\partial x}(t,x) = M(x)y(t,x) + Ju(t,x), \quad \mathrm{rank}\, J < n?$$

- Nonlocal boundary conditions : [1]

$$\begin{pmatrix} y_-(t,1) \\ y_+(t,0) \end{pmatrix} = Q \begin{pmatrix} y_-(t,0) \\ y_+(t,1) \end{pmatrix}?$$

- $\mathrm{supp}\, u_k \subset (0,T) \times \omega_k$ with ω_k **disjoints** ?
- Null controllability ?
- Moving control domain $\omega = \omega(t)$?
- Quasi-linear systems ? [1] [2]

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Thank you for your attention !

More details available at :
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