On the controllability of sub-elliptic systems of Grushin type

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The system of interest

Let $T > 0$,

$$\begin{cases} \partial_t f - \Delta_x f - |x|^{2\gamma}\Delta_y f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f & = & 0, & (t,x,y) \in (0,T)\times\Omega, \\ f(t,x,y) & = & 0, & (t,x,y) \in (0,T)\times\partial\Omega, \\ f(0,x,y) & = & f_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{1}$$

where

- $\Omega = \Omega_x \times \Omega_y$, with $0_{\mathbb{R}^{d_x}} \in \Omega_x \subset \mathbb{R}^{d_x}$, and Ω_y is a compact Riemannian manifold of dimension d_y, with metric σ and volume form dvol_σ,
- Δ_y is the Laplace-Beltrami operator on Ω_y,
- $f_0 \in L^2(\Omega, dx\,\mathrm{dvol}_\sigma)$,
- $\gamma \geq 1$ and $\nu > 0$,
- H is the Hardy constant that depends on the geometric setting for Ω_x (*i.e.* best constant such that),

$$\mathrm{H}\int_{\Omega_x} \frac{|u|^2}{|x|^2}\,dx \leq \int_{\Omega_x} |\nabla_x u|^2\,dx, \quad \text{for every } u \in H_0^1(\Omega_x). \tag{2}$$

Let $T > 0$,

$$\begin{cases} \partial_t f - \Delta_x f - |x|^{2\gamma} \Delta_y f + \frac{\nu^2 - \mathrm{H}}{|x|^2} f & = & 0, & (t,x,y) \in (0,T) \times \Omega, \\ f(t,x,y) & = & 0, & (t,x,y) \in (0,T) \times \partial\Omega, \\ f(0,x,y) & = & f_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{3}$$

Definition

We say that system (3) is observable in time $T > 0$ from $\omega \subset \Omega$ if there exists a constant $C > 0$ such that, for every $f_0 \in L^2(\Omega, dx\, \mathrm{dvol}_\sigma)$, the associated solution f of system (3) satisfies

$$\int_\Omega |f(T,x,y)|^2 \, dx \; \mathrm{dvol}_\sigma \leq C \int_0^T \int_\omega |f(t,x,y)|^2 \, dx \; \mathrm{dvol}_\sigma \, dt. \tag{4}$$

$\Longleftrightarrow$ null-controllability: for every $f_0 \in L^2(\Omega)$, there exists $u \in L^2((0,T) \times \Omega)$ such that the solution of $\left(\partial_t - \Delta_x - |x|^{2\gamma}\Delta_y + \frac{\nu^2 - \mathrm{H}}{|x|^2}\right) f = \mathbf{1}_\omega u$ satisfies $f(T) = 0$.

Outline of the talk

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Motivation: a general subelliptic system

- $\mathcal{M}$ is a (smooth, connected, oriented) compact n-dimensional manifold.
- $X = (X_1, ..., X_m)$ is an m-tuple of vector fields on $\mathcal{M}$, not necessarily linearly independent.
- Denoting $D^1 = \text{Span}(X)$, $D^{k+1} = D^k + [D, D^{k-1}]$, there exists k such that $D^k(p) = T_p\mathcal{M}$.
- μ is a smooth measure on $\mathcal{M}$.
- Δ is the associated sub-Laplacian

$$\Delta = \sum_{i=1}^{m} X_i^* X_i, \tag{5}$$

where X_i^* is the formal adjoint of X_i in $L^2(\mathcal{M}, \mu)$.

We are motivated by the study of the observability properties of

$$\begin{cases} \partial_t f - \Delta f = 0, & (t,p) \in (0,T) \times \mathcal{M}, \\ f(t,p) = 0, & (t,p) \in (0,T) \times \partial\mathcal{M}, \text{ if } \partial\mathcal{M} \neq \emptyset, \\ f(0,p) = f_0(p), & p \in \mathcal{M}, \end{cases} \tag{6}$$

when $D^1(p) \neq T_p\mathcal{M}$ for some $p \in \mathcal{M}$.

Motivation: Grushin as a toy model

- $\Omega = (0, L) \times \Omega_y$, for some $L > 0$ and Ω_y is a compact Riemannian manifold.
- On Ω, we consider the (sub-Riemannian) metric $dx^2 + x^{-2\gamma}\sigma$, where σ is a Riemannian metric on Ω_y.

The associated Laplace-Beltrami (sub-Laplacian for the measure taken as the Popp's measure associated to the metric) operator is given by

$$\Delta_\gamma = -\partial_x^2 - x^{2\gamma}\Delta_y + \frac{\gamma d_y}{x}\partial_x f, \tag{7}$$

where Δ_y is the Laplace-Beltrami on Ω_y, and Δ_γ acts on $L^2(\Omega, x^{-\gamma d_y} dx \, \mathrm{dvol}_\sigma)$.The change of variable $f = x^{\gamma d_y/2} g$ leads to

$$G_\gamma g = -\partial_x^2 g - x^{2\gamma}\Delta_y g + \frac{\gamma d_y}{2}\left(\frac{\gamma d_y}{2} + 1\right)\frac{g}{x^2}, \tag{8}$$

that acts on $L^2(\Omega, dx \, \mathrm{dvol}_\sigma)$.

Our system generalizes the dimension in x, and the considerations for the singular term.

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Some known results of observability for $\gamma = 1$

$$\begin{cases} \partial_t f - \Delta_x f - |x|^2 \Delta_y f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f = 0, & (t,x,y) \in (0,T) \times \Omega, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega, \\ f(0,x,y) = f_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{9}$$

Theorem (P. Cannarsa - R. Guglielmi, 2014)

Consider system (9) with $\Omega = (0,1) \times (0,1)$, and $\mathrm{H} = 1/4$ is the Hardy constant.

For every $\nu > 0$, for every $\omega = (a,b) \times (0,1)$, $0 < a < b \leq 1$, there exists a time $T^* > 0$ such that system (9) is observable from ω in any time $T > T^*$.

Theorem (CT. Anh - VM. Toi, 2016)

Consider system (9) with $\Omega = \Omega_x \times \Omega_y \subset \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ a bounded domain such that $d_x \geq 3$, $d_y \geq 1$, $0_{\mathbb{R}^{d_x}} \in \Omega_x$, and $H = (d_x - 2)^2/4$ is the Hardy constant.

For every $\nu > 0$, for every $\omega = \omega_x \times \Omega_y$ such that $0_{\mathbb{R}^{d_x}} \notin \omega_x$, there exists a time $T^* > 0$ such that system (9) is observable from ω in any time $T > T^*$.

A unique continuation result

Set $\Omega = (-1,1) \times (0,1)$. Let $T > 0$,

$$\begin{cases} \partial_t f - \partial_x^2 f - x^{2\gamma}\partial_y^2 f + \dfrac{\nu^2 - 1/4}{x^2} f = 0, & (t,x,y) \in (0,T) \times \Omega, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega, \\ f(0,x,y) = f_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{10}$$

NB: particular choice of domain for $-\partial_x^2 - x^{2\gamma}\partial_y^2 + \dfrac{\nu^2 - 1/4}{x^2}$ (transmission conditions across $\{x = 0\}$).

Theorem (M. Morancey, 2015)

Consider system (10) with $\omega \subset \Omega$ and $\nu \in (0,1)$. If the solution f of (10) with $f_0 \in L^2(\Omega)$ vanishes on $\omega \times (0,T)$, then $f_0 = 0$.

Our results

Recall our system of interest,

$$
\begin{cases}
\partial_t f - \Delta_x f - |x|^{2\gamma} \Delta_y f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f & = & 0, & (t, x, y) \in (0, T) \times \Omega, \\
f(t, x, y) & = & 0, & (t, x, y) \in (0, T) \times \partial\Omega, \\
f(0, x, y) & = & f_0(x, y), & (x, y) \in \Omega.
\end{cases} \tag{11}
$$

We assume that

(H_1) if $d_x = 1$, then $\Omega_x = (0, L)$ for some $L > 0$,

(H_2) if $d_x \geq 3$, then $0 \in \Omega_x$.

We denote by $T(\omega)$ the minimal time observability from ω.

Our results

$$\begin{cases} \partial_t f - \Delta_x f - |x|^{2\gamma}\Delta_y f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f = 0, \\ \text{Dirichlet boundary conditions,} \\ f_0 \in L^2(\Omega). \end{cases}$$

(H_1) if $d_x = 1$, then $\Omega_x = (0, L)$,

(H_2) if $d_x \geq 3$, then $0 \in \Omega_x$.

Theorem (V., 2025)

Let $\gamma = 1$. For every $\nu > 0$ and $\omega = \omega_x \times \Omega_y \subset \Omega$ an open set,

$$T(\omega) = \frac{\operatorname{dist}(\omega_x, 0)^2}{4(1+\nu)}. \tag{12}$$

Theorem (V., 2025)

Let $\gamma > 1$. For every $\nu > 0$ and $\omega \subset \Omega$ an open set such that $0 \notin \overline{\omega}$,

$$T(\omega) = +\infty. \tag{13}$$

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Sketch of proofs: Fourier decomposition

$$\begin{cases} \partial_t f - \Delta_x f - |x|^{2\gamma}\Delta_y f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f &= 0, & (t,x,y) \in (0,T)\times\Omega, \\ f(t,x,y) &= 0, & (t,x,y)\in(0,T)\times\partial\Omega, \\ f(0,x,y) &= f_0(x,y), & (x,y)\in\Omega. \end{cases}$$

$\downarrow$ Fourier decomposition in y, $f(t,x,y) = \sum_{n\geq 1} f_n(t,x)\phi_n(y)$

$$\begin{cases} \partial_t f_n - \Delta_x f_n + \xi_n^2|x|^{2\gamma} f_n + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f_n &= 0, & (t,x) \in (0,T)\times\Omega_x, \\ f_n(t,x) &= 0, & (t,x)\in(0,T)\times\partial\Omega_x, \\ f_n(0,x) &= f_{0,n}(x,y), & x\in\Omega_x. \end{cases}$$

Sketch of proofs: uniform observability

Let $T > 0$, and $\omega = \omega_x \times \Omega_y$. There exists $C > 0$ such that for every $f_0 \in L^2(\Omega)$, the associated solution satisfies

$$\int_\Omega |f(T,x,y)|^2 \, dx \; \mathrm{dvol}_\sigma \leq C \int_0^T \int_\omega |f(t,x,y)|^2 \, dx \; \mathrm{dvol}_\sigma \, dt. \tag{14}$$

$$\Updownarrow \quad \text{Fourier decomposition in } y$$

Let $T > 0$. There exists $C > 0$ such that for every $n \geq 1$, for every $f_{0,n} \in L^2(\Omega_x)$, the associated solution satisfies

$$\int_{\Omega_x} |f_n(T,x)|^2 \, dx \leq C \int_0^T \int_{\omega_x} |f(t,x)|^2 \, dx \, dt. \tag{15}$$

We therefore study the *uniform observability*.

Sketch of proofs: uniform observability

Study of the existence of a uniform cost $C > 0$ of observability with respect to n,

$$\int_{\Omega_x} |f_n(T,x)|^2 \, dx \leq C \int_0^T \int_{\omega_x} |f_n(t,x)|^2 \, dx \, dt,$$

for the solutions of

$$\begin{cases} \partial_t f_n - \Delta_x f_n + \xi_n^2 |x|^{2\gamma} f_n + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f_n & = & 0, & (t,x) \in (0,T) \times \Omega_x, \\ f_n(t,x) & = & 0, & (t,x) \in (0,T) \times \partial\Omega_x, \\ f_n(0,x) & = & f_{0,n}(x,y), & x \in \Omega_x. \end{cases}$$

Sketch of proofs: uniform observability for $\gamma = 1$

We use an interplay between the cost of small time observability and dissipation speed, for large n.

Let $T_0 > 0$, there exists $C, d > 0$ such that

$$\int_{\Omega_x} |f_n(T_0, x)|^2 \, dx \leq C e^{d\xi_n} \int_0^{T_0} \int_{\omega_x} |f_n(t, x)|^2 \, dx \, dt.$$

Let $T > T_0 > 0$, the dissipation speed is

$$\int_{\Omega_x} |f_n(T, x)|^2 \, dx \leq e^{-4\xi_n(1+\nu)(T - T_0)} \int_{\Omega_x} |f_n(T_0, x)|^2 \, dx.$$

The dissipation speed beats the cost of small time observability as long as

$$T \geq \frac{d}{4(1+\nu)} + T_0.$$

The constant d depends on ω_x and arbitrary small parameters, and we assume first that $0 \notin \omega_x$. The case $0 \in \omega_x$ then follows as a limit-case.

Sketch of proofs: observability via Carleman estimates

Carleman estimates for solutions of

$$\begin{cases} \partial_t f - \Delta_x f + \xi^2 |x|^{2\gamma} f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f & = & \mathrm{F}, & (t,x) \in (0,T) \times \Omega_x, \\ f(t,x) & = & 0, & (t,x) \in (0,T) \times \partial\Omega_x, \\ f(0,x) & = & f_0(x,y), & x \in \Omega_x. \end{cases}$$

We observe that the Carleman weight of [K. Beauchard - J. Dardé - S. Ervedoza '20] in the non-singular case is well suited for our operator

$$\varphi_\xi(t,x) = \frac{\xi}{2}(L^2 - |x|^2)\coth(2\xi t), \quad \text{where } L = \sup\{|x|,\ x \in \Omega_x\}.$$

Two reasons:

(i) Inspired from the heat kernel of the non-singular operator on the real line, which resembles very much the one of the singular operator on the half-line (both show roughly the same dynamic).

(ii) Spatial part of the weight is well-suited to deal with inverse square potentials for the heat equation (see e.g. [S. Ervedoza '08])

Sketch of proofs: observability via Carleman estimates

$$\begin{cases} \partial_t f - \Delta_x f + \xi^2 |x|^{2\gamma} f + \dfrac{\nu^2 - \mathrm{H}}{|x|^2} f & = & \mathrm{F}, & (t,x) \in (0,T) \times \Omega_x, \\ f(t,x) & = & 0, & (t,x) \in (0,T) \times \partial\Omega_x, \\ f(0,x) & = & f_0(x,y), & x \in \Omega_x. \end{cases} \tag{16}$$

$$\varphi_\xi(t,x) = \frac{\xi}{2}(L^2 - |x|^2)\coth(2\xi t), \quad \text{where } L = \sup\{|x|,\ x \in \Omega_x\}. \tag{17}$$

Proposition (Boundary Carleman estimate)

Let φ_ξ be defined by (17). For any solution f of system (16), the function $g = fe^{-\varphi_\xi}$ satisfies

$$\int_{\Omega_x} |\nabla_x g(T,x)|^2 - \frac{\xi^2 L^2}{\sinh(2\xi t)^2}|g(T,x)|^2 + \frac{\nu^2 - \mathrm{H}_{d_x}}{|x|^2}|g(T,x)|^2 \, dx$$
$$\leq \int_0^T \int_{\Omega_x} |\mathrm{F}|^2 e^{-2\varphi_\xi} \, dx \, dt + \xi L \int_0^T \frac{\sinh(4\xi t)}{\sinh(2\xi t)^2} \int_{\Gamma_+} |\nabla_x g \cdot \eta|^2 \, dS \, dt, \tag{18}$$

where $\Gamma_+ = \{x \in \partial\Omega_x, x \cdot \eta > 0\}$, and η is the normal outward pointing unit vector at the boundary.

Proposition (Internal observability)

Let $x_0 \in \Omega_x \setminus \{0\}$. *For any* $\epsilon, \epsilon' > 0$ *such that* $0 \notin B(x_0, \epsilon) \subset\subset \Omega_x$, *there exists* $C > 0$, $\xi_0 > 0$, *such that for every* $\xi \geq \xi_0$, *for every* u_ξ *solution of system* (16) *with* $f_\xi = 0$, *we have*

$$\int_{\Omega_x} |u_\xi(T,x)|^2 \, dx \leq C\xi^2 e^{\xi(1+\epsilon')M} \int_0^T \int_{B(x_0,\epsilon)} |u_\xi(t,x)|^2 \, dx \, dt, \tag{19}$$

where $M := \max\{|x|^2, x \in B(x_0, \epsilon)\}$.

Sketch of proofs: non-observability

We assume that $0 \notin \omega_x$. We use an interplay between exponential decay of eigenfunctions and dissipation speed.

Denote by u_ξ the first eigenfunction of

$$-\Delta_x f + \xi^2 |x|^{2\gamma} f + \frac{\nu^2 - \mathrm{H}}{|x|^2} f = \lambda_\xi u_\xi.$$

For any $\delta > 0$ sufficiently small, and $\xi > 0$ sufficiently large,

$$\int_{\omega_x} u_\xi(x)^2 \, dx \leq C e^{-d(\delta)\xi(1-\delta)}, \quad d(\delta) \underset{\delta \to 0^+}{\longrightarrow} \mathrm{dist}(\omega_x, 0)^2.$$

Let $T > 0$, the dissipation speed for $u_\xi(t, \cdot)$ is

$$\int_{\Omega_x} |u_\xi(T,x)|^2 \, dx \geq \begin{cases} e^{(-4\xi(1+\nu)+o(\xi))T} \int_{\Omega_x} |u_\xi(x)|^2 \, dx & \text{if } \gamma = 1, \\ e^{-2c\xi^{2/(\gamma+1)}T} \int_{\Omega_x} |u_\xi(x)|^2 \, dx & \text{for some } c > 0 \text{ if } \gamma \geq 1. \end{cases}$$

Sketch of proofs: non-observability

$$\int_{\omega_x} u_\xi(x)^2 \, dx \leq C e^{-2d(\delta)\xi(1-\delta)}, \quad d(\delta) \underset{\delta \to 0^+}{\longrightarrow} \mathrm{dist}(\omega_x, 0)^2. \tag{ED}$$

$$\int_{\Omega_x} |u_\xi(T,x)|^2 \, dx \geq \begin{cases} e^{(-4\xi(1+\nu)+o(\xi))T} \int_{\Omega_x} |u_\xi(x)|^2 \, dx & \text{if } \gamma = 1, \\ e^{-2c\xi^{2/(\gamma+1)}T} \int_{\Omega_x} |u_\xi(x)|^2 \, dx & \text{for some } c > 0 \text{ if } \gamma \geq 1. \end{cases} \tag{DS}$$

We test the uniform observability against $e^{-\lambda_\xi t} u_\xi$:

$$e^{-2\lambda_\xi T} \leq C \int_{\omega_x} |u_\xi(x)|^2 \, dx. \tag{20}$$

When $\gamma = 1$, in (20), the dissipation speed cannot beat the exponential decay as $\xi \to +\infty$ if

$$T \leq (1-\delta) \frac{d(\delta)}{4(1+\nu)}.$$

When $\gamma > 1$, the exponential decay is stronger than the dissipation as $\xi \to +\infty$, in any time. So (20) never holds.

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Comparison with the non-singular Grushin equation ($\nu^2 = \mathrm{H}$, $\gamma = 1$)

$$\begin{cases} \partial_t f - \Delta_x f - |x|^2 \Delta_y f = 0, & (t,x,y) \in (0,T) \times \Omega, \\ f(t,x,y) = 0, & (t,x,y) \in (0,T) \times \partial\Omega, \\ f(0,x,y) = f_0(x,y), & (x,y) \in \Omega. \end{cases} \tag{21}$$

Theorem (K. Beauchard - J. Dardé - S. Ervedoza, 2020)

(i) In the case $\Omega_x = (-L_-, L_+)$ for some $L_-, L_+ > 0$, the minimal of time (boundary) observability of system (21) is

$$T\left((\{-L_-\} \cup \{L_+\}) \times \Omega_y\right) = \min\left(\frac{L_-^2}{2}, \frac{L_+^2}{2}\right). \tag{22}$$

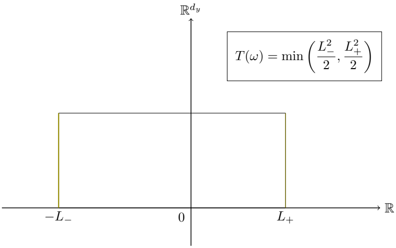
(ii) In the case $\Omega_x = B(0,L) \subset \mathbb{R}^{d_x}$, the minimal of time (boundary) observability of system (21) is

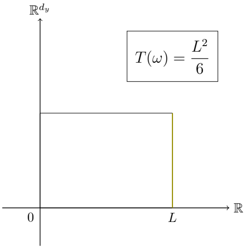
$$T\left(\partial\Omega_x \times \Omega_y\right) = \frac{L^2}{2d_x}. \tag{23}$$

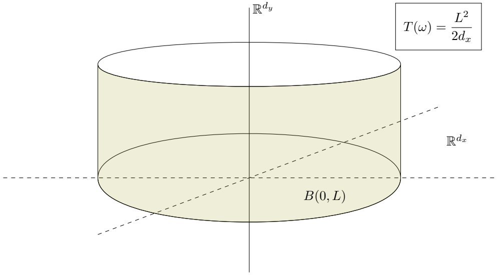
(iii) In the case $\Omega_x = (0,L)$ for some $L > 0$, the minimal of time (boundary) observability of system (21) is

$$T\left(\{L\} \times \Omega_y\right) = \frac{L^2}{6}. \tag{24}$$

Comparison with the non-singular Grushin equation ($\nu^2 = \mathrm{H}$)







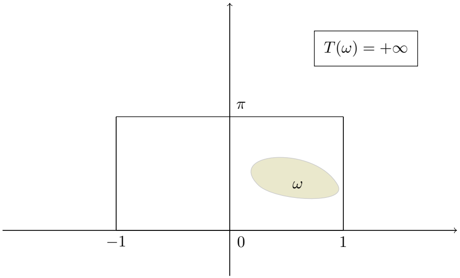
Comparison with the non-singular Grushin equation ($\nu^2 = \mathrm{H}$, $\gamma > 1$)

Set $\Omega = (-1,1)\times(0,\pi)$. Let $T > 0$,

$$\begin{cases} \partial_t f - \Delta_x f - x^{2\gamma}\Delta_y f = 0, & (t,x,y)\in(0,T)\times\Omega, \\ f(t,x,y) = 0, & (t,x,y)\in(0,T)\times\partial\Omega, \\ f(0,x,y) = f_0(x,y), & (x,y)\in\Omega, \end{cases} \tag{25}$$

Theorem (K. Beauchard - P. Cannarsa - R. Guglielmi (2012))

Let $\gamma > 1$. For every $\omega \subset \Omega$ such that $(\{0\}\times(0,\pi))\cap\overline{\omega} = \emptyset$, system (25) is never observable, *i.e.* $T(\omega) = +\infty$.



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The previous non-singular equations correspond to the sub-Laplacian associated to $X_i = \partial_{x_i}$, $Y_j = |x|^\gamma \partial_{y_j}$, and the Lebesgue measure. Namely,

$$-\partial_x^2 f - |x|^{2\gamma} \Delta_y f$$

The Laplace-Beltrami operator ($d_x = 1$ and $d_y \geq 1$) will correspond to our singular operator with $\nu^2 = (\gamma d_y + 1)^2/4$. Namely,

$$-\partial_x^2 f - |x|^{2\gamma} \Delta_y f + \frac{\gamma d_y}{2} \left(\frac{\gamma d_y}{2} + 1 \right) \frac{f}{x^2}.$$

When $\gamma = 1$, the minimal time of observability depends on,

- in the non-singular case $\rightarrow$ dependence on d_x but not on d_y,
- in the singular case $\rightarrow$ dependence on $\nu > 0$ but not (explicitly) on d_x, and if $d_x = 1$ and $\nu^2 = (\gamma d_y + 1)^2/4$, we have dependence on d_y since

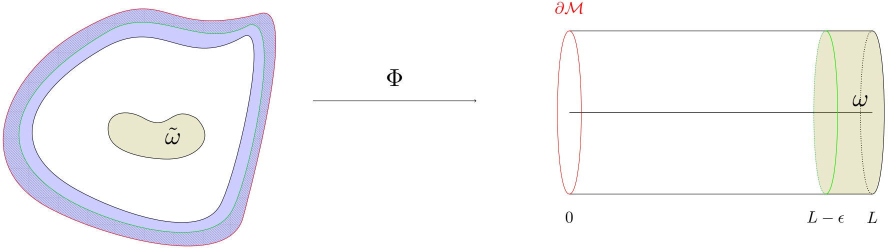
$$T(\omega) = \frac{\mathrm{dist}(\omega_x, 0)^2}{4(1+\nu)} = \frac{\mathrm{dist}(\omega_x, 0)^2}{6 + 2d_y}.$$

Some comments and applications

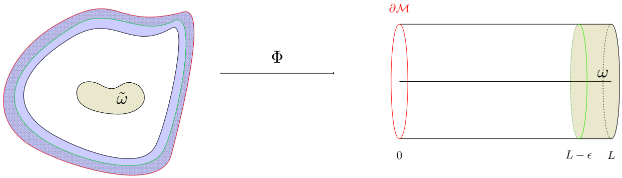
We can infer a result on compact $(n+1)$-dimensional manifolds. Let $T > 0$,

$$\begin{cases} \partial_t f - \Delta f &= 0, & (t,p) \in (0,T) \times \mathcal{M}, \\ f(t,p) &= 0, & (t,p) \in (0,T) \times \partial\mathcal{M}, \text{ if } \partial\mathcal{M} \neq \emptyset, \\ f(0,p) &= f_0(p), & p \in \mathcal{M}. \end{cases} \tag{26}$$

We use a *reduction process*



Some comments and applications



Under suitable assumptions, the blue neighborhood $\mathcal{U}$ of the boundary $\partial\mathcal{M}$ in red can be constructed to be diffeomorphic to $[0, L) \times \partial\mathcal{M}$, and such that in coordinates, the metric is of the form

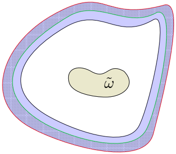
$$dx^2 + x^{-\gamma}\sigma(x),$$

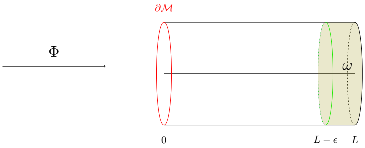
where $\sigma(x)$ is a family of smooth Riemannian metrics on $\partial\mathcal{M}$ that depends continuously on x.

We assume that $\sigma(x) = \sigma$ in $\mathcal{U}$, so that in coordinates we recover (a particular case of) our singular Grushin operator,

$$-\partial_x^2 - |x|^{2\gamma}\Delta_y + \frac{\gamma d_y}{2}\left(\frac{\gamma d_y}{2} + 1\right)\frac{\text{Id}}{x^2}. \quad (27)$$

Some comments and applications





In $\mathcal{U}$ (in blue), the metric is $dx^2 + x^{-\gamma}\sigma$

Theorem (V., 2025)

Under some some suitable geometric assumptions, we have the following lower bounds if $\overline{\omega} \cap \partial\mathcal{M} = \emptyset$.

(i) If $\gamma = 1$, there exists $L > 0$ such that the (possibly infinite) minimal time of observability satisfies

$$T(\omega) \geq \frac{L^2}{6+2n}. \quad (28)$$

(ii) If $\gamma > 1$, we have $T(\omega) = +\infty$.

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The lower bound $\frac{L^2}{6+2n}$ is sharp for some well-chosen ω.

If one asks the measure μ on $\mathcal{M}$ to write in $\mathcal{U}$ like $\mu = dx\,\mathrm{dvol}_\sigma$, then the lower bound becomes

$$T(\omega) \geq \frac{L^2}{6},$$

which is also sharp for some well-chosen ω.

Thank you for your attention!

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